Math 355, Winter 2011 D. Jakobson

Hausdorff Dimension Problem Set, Part 1
All problems are extra credit

Do all the problems. Every problem is worth 5 points. This problem set follows the exposition in Falconer's book *Fractal Geometry: Mathematical Foundations and Applications.*

Let $s \geq 0$. The Hausdorff s-dimensional measure of a set $F \in \mathbf{R}^n$ is defined as follows: Let $\{U_i\}_{i \in I}$ be a (countable or finite) cover of F by sets of diameter at most $\delta > 0$. Let $|U| = \operatorname{diam}(U) = \sup_{x,y \in U} |x - y|$. We let

$$\mathcal{H}^s_\delta(F) := \inf\{\sum_{i \in I} |U_i|^s\}, \tag{1}$$

where the infimum is taken over all covers of F by sets diameter at most δ. It is clear that $\mathcal{H}^s_\delta(F)$ increases as $\delta \to 0$.

We next define the Hausdorff s-dimensional $\mathcal{H}^s(F)$ by

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(F). \tag{2}$$

The limit exists for any subset $F \in \mathbf{R}^n$ and is usually equal to 0 or ∞. It is clear that $\mathcal{H}^s(\emptyset) = 0$, and that $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$ for $E \subset F$.

Problem 1. Let $\{F_j\}$ be a countable collection of disjoint Borel sets. Show that

$$\mathcal{H}^s(\cup_j F_j) = \sum_j \mathcal{H}^s(F_j).$$

One can also show that for $s = n$, $\mathcal{H}^n$ is just a constant multiple of the n-dimensional volume in $\mathbf{R}^n$ (can you figure out the value of the constant?) Also, $\mathcal{H}^0$ is just the counting measure.

Problem 2.

i) For $\lambda > 0$, let $\lambda F := \{\lambda x : x \in F\}$. Show that

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F).$$

ii) Let $f : F \to \mathbf{R}^m$ be a mapping satisfying

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \qquad x, y \in F$$

for some constant $c > 0, \alpha > 0$ (Hölder with exponent α). Show that

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F).$$

What happens for $\alpha = 1$ (Lipschitz maps)?

It is clear that for a fixed $F \in \mathbf{R}^n$ and $\delta < 1$, the function $\mathcal{H}^s_\delta(F)$ is a non-increasing function of s.

Problem 3. Show that if $\mathcal{H}^s(F) < \infty$, then

$$\mathcal{H}^t(F) = 0, \qquad \forall t > s.$$

Conclude that there exists at most one value of s for which $0 < \mathcal{H}^s(F) < \infty$.

It follows from Problem 3 that there exists a unique value of $s_0 \geq 0$ such that $\mathcal{H}^s(F) = \infty, s < s_0$ and $\mathcal{H}^s(F) = 0, s > s_0$. We call s_0 the *Hausdorff dimension* of F,

$$\dim_H F = s_0.$$

The measure $\mathcal{H}^{s_0}(F)$ can be finite or equal to 0 or ∞.