McGill University

Math 355: Honors Analysis 4 Winter 2011

Fourier Analysis Handout

This is a summary of some Material in Section 2, Chapter 1, of Katznelson's *Introduction to Harmonic Analysis.*

Lemma 1. Let $f \in L^1(\mathbf{T})$, and let $f_\tau(t) := f(t-\tau)$. Then

$$\lim_{\tau \to 0} ||f - f_\tau||_1 = 0. \tag{1}$$

Proof: For any $f \in L^1(\mathbf{T})$ and any $\epsilon > 0$, there exists $g \in C(\mathbf{T})$ s.t. $||f-g||_1 < \epsilon$. We have

$$||f - f_\tau||_1 \leq ||f-g||_1 + ||g - g_\tau||_1 + ||f_\tau - g_\tau||_1 \leq ||g - g_\tau||_1 + 2\epsilon.$$

Now, a continuous function on $\mathbf{T}$ is uniformly continuous, so (1) holds for g, and we find that $\limsup_{\tau \to 0} ||f - f_\tau||_1 \leq 2\epsilon$. Since ϵ was arbitrary, we have proved Lemma 1.

QED

Lemma 2. Let $\{k_n(\tau)\}$ be a summability kernel satisfying the properties mentioned in class, and let $\phi : \mathbf{T} \to X$ be a continuous function into a Banach space X. Then

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{\mathbf{T}} k_n(\tau)\phi(\tau) d\tau = \phi(0).$$

Here the equality is between elements of X.

Proof: We have

$$\frac{1}{2\pi} \int_{\mathbf{T}} k_n(\tau)\phi(\tau) d\tau - \phi(0) = \frac{1}{2\pi} \int_{\mathbf{T}} k_n(\tau)(\phi(\tau) - \phi(0)) d\tau.$$

We can rewrite that as

$$\frac{1}{2\pi} \int_{\delta}^{\delta} k_n(\tau)(\phi(\tau) - \phi(0)) d\tau + \frac{1}{2\pi} \int_{\delta}^{2\pi - \delta} k_n(\tau)(\phi(\tau) - \phi(0)) d\tau := I_1 + I_2.$$

The first summand can be estimated from above by

$$||I_1||_X \leq \max_{|\tau| \leq \delta} ||\phi(\tau) - \phi(0)||_X \cdot ||k_n||_{L^1}.$$

Since $||k_n||_{L^1} \leq C$ uniformly and ϕ is continuous at $\tau = 0$, given $\epsilon > 0$, we can find $\delta > 0$ s.t. $||I_1||_X < C \cdot \epsilon$.

To estimate I_2, we note that

$$||I_2||_X \leq \max_{\tau} ||\phi(\tau) - \phi(0)||_X \cdot \frac{1}{2\pi} \int_{\delta}^{2\pi - \delta} |k_n(\tau)| d\tau \leq C_1 \int_{\delta}^{2\pi - \delta} |k_n(\tau)| d\tau.$$

Since the last integral goes to 0 as $n \to \infty$ by the assumption on k_n, we find that (for δ as above), $\lim_{n \to \infty} ||I_1 + I_2||_X \leq C \cdot \epsilon$.

QED.

Lemma 1 and Lemma 2 immediately imply

Theorem 3. Let $f \in L^1(\mathbf{T})$, and $\{k_n\}$ a summability kernel. Then

$$f = \lim_{n \to \infty} \frac{1}{2\pi} \int_{\mathbf{T}} k_n(\tau) f_\tau d\tau = \lim_{n \to \infty} k_n * f.$$

The last equality is first proved for continuous f by approximating the convolution integral by a sequence of Riemann sums; the statement for general $f \in L^1$ follows by approximation.