McGill University
Math 354: Honors Real Analysis 3

Definition. An algebra $\mathcal{A}$ of functions is called *separating* if for any $x \neq y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Suppose $\mathcal{A}$ is unital (contains the constant function $\mathbf{1}$) and separating. Given two numbers $a, b \in \mathbf{R}$, let $d = (a-b)/(f(x)-f(y))$ and let $c = a - d$. Then the function $g = c \cdot \mathbf{1} + d \cdot f$ satisfies $g(x) = a, g(y) = b$.
Theorem (Stone, Wierstrass). Let K be compact, and let $\mathcal{A}$ be unital separating subalgebra of $C(K)$. Then $\mathcal{A}$ is dense in $C(K)$ (with uniform metric).
Proof. (S. Drury's Math 354 notes, Theorem 97, p. 119). The proof uses the following
Lemma 1. The closure $\overline{\mathcal{A}} \subset C(K)$ is also a unital separating subalgebra of $C(K)$.

To prove Lemma 1, we note that $\overline{\mathcal{A}}$ is clearly unital ($\mathbf{1} \in \mathcal{A} \subset \overline{\mathcal{A}}$) and separating, since $\mathcal{A}$ is. To prove that it's a subalgebra, let $f_n \to f, g_n \to g$ uniformly. Then it follows from an easy application of the triangle inequality that

$$||f_n \cdot g_n - f \cdot g|| = ||f_n g_n - f_n g + f_n g - fg|| \leq ||f - f_n|| \cdot ||g|| + ||f_n|| \cdot ||g - g_n|| \to 0$$

as $n \to \infty$, so $f_n g_n \to fg$ uniformly, and hence $fg \in \overline{\mathcal{A}}$, QED. Accordingly, it suffices to assume that $\mathcal{A} = \overline{\mathcal{A}}$ is closed, and prove that then $\mathcal{A} = C(K)$.
Lemma 2. For any $C > 0$, there exists a sequence p_n of real polynomials that converge uniformly to $|x|$ on $[-C, C]$. For the proof, rescale to the interval $[0, 1]$ and use Bernstein approximation theorem, or see Lemma 100 in S. Drury's Math 354 notes.
Lemma 3. If $f, g \in \mathcal{A}$, then $\max(f, g) \in \mathcal{A}$ and $\min(f, g) \in \mathcal{A}$. For the proof, note that $|f - g| \in \mathcal{A}$ by Lemma 2 and the assumption that $\mathcal{A}$ is closed, since all polynomials $p_n(h)$ in $h = f - g$ belong to $\mathcal{A}$. Next, note that $\max(f, g) = (f + g + |f - g|)/2$ while $\min(f, g) = (f + g - |f - g|)/2$.
Proof of the Theorem. Let $f \in C(K)$ and let $\epsilon > 0$. Let $x \in K$ (which will be fixed for a moment). Let $x \neq y \in K$. Since $\mathcal{A}$ is separating, there exists $h_{x,y} \in \mathcal{A}$ such that $h_{x,y}(x) = f(x), h_{x,y}(y) = f(y)$. By continuity at x, there exists a neighborhood $V_{x,y}$ of x such that $h_{x,y}(z) - f(z) < \epsilon$ for $z \in V_{x,y}$. We have (for x fixed!) $K = \cup_{y \in K} V_{x,y}$, and by compactness $K = \cup_{k=1}^{m} V_{x,y_k}$, where y_j-s depend on x. Let

$$g_x = \min_k h_{x,y_k}.$$

Then $g_x \in \mathcal{A}$ by Lemma 3 since $\mathcal{A}$ is closed. Also, $g_x(x) = f(x)$, and for any $z \in K$, $z \in V_{x,y_k}$ for some $1 \leq k \leq m$, so $g_x(z) \leq h_{x,y_k}(z) < f(z) + \epsilon$.

Next, by continuity for any $x \in K$ there exists an open neighborhood U_x of x such that $g_x(z) > f(z) - \epsilon$ for all $z \in U_x$. We have $K = \cup_{x \in K} U_x$, and by compactness $K = \cup_{j=1}^{l} U_{x_j}$. Let

$$g = \max_j g_{x_j}.$$

Then $g \in \mathcal{A}$ by Lemma 3 as before, and for any $z \in K$, we have $z \in V_{x_j}$ for some $1 \leq j \leq l$, so $g(z) \geq g_{x_j}(z) > f(z) - \epsilon$. Also, $g(z) < f(z) + \epsilon$ since all $g_x(z)$ satisfy that inequality. It follows that $f(z) - \epsilon < g(z) < f(z) + \epsilon$. Since ϵ and f were arbitrary, the proof is finished.

QED