Math 354: Analysis 3
Assignment 6

Nicholas Leavitt
260480731

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2. Let $\delta > 0$ be given, then there exists a continuous function g of compact support so that $\|f(x)-g(x)\| < \epsilon$, then since $f(x)-f(\delta x) = f(x)-g(x)+g(x)-g(\delta x)+g(\delta x)-g(x)$. But $\|f(\delta x)-g(\delta x)\| = \delta^d\|f(x)-g(x)\| \to \epsilon$ with $\delta \to 1$ and $\|g(\delta x)-g(x)\| \to 0$ since $g(x)$ is continuous and of compact support, whence $\|f(\delta x)-f(x)\| \leq 3\epsilon$ as desired.

5. a) For any $z, w \in F$ we have,

$$||x-z|-|y-w|| \leq |x-z-y+w| \leq |x-y|+|w-z| \leq |x-y|$$

in particular, this holds for any sequence of points $\{z_n\}$ such that $|x-z_n| \to \delta(x)$, similarly with $\{w_n\}$ so that,

$$|\delta(x)-\delta(y)| = \lim_{n\to\infty} |||x-z_n|-|y-w_n|| \leq |x-y|$$

as desired.

b) First note that as $x \notin F$ and F^C is closed, there exists an open neighborhood $B(x;\epsilon)$ $(\epsilon > 0)$ of x contained entirely in F^C. Now consider $N = B(x;\epsilon/2)$, for any $y \in N$ we have $\delta(y) \geq \epsilon/2$ since $|y-w| \geq \epsilon/2$ by construction for all $w \in F$. Now,

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2}dy \geq \int_N \frac{\delta(y)}{|x-y|^2}dy \geq \frac{\epsilon}{2}\int_N \frac{1}{|x-y|^2}dy = \infty$$

whence $I(x) = \infty$ for $x \notin F$.

c) We have by Fubini's theorem, as $\delta(y)/|x-y|^2 \geq 0$ and $\delta(y) = 0$ for $y \in F$,

$$\int_F I(x)dx = \int_F\int_{F^C} \frac{\delta(y)}{|x-y|^2}dydx = \int_{F^C}\delta(y)\int_F \frac{1}{|x-y|^2}dxdy$$

for arbitrary fixed $y \notin F$, we have for some constant ϵ_y, $0 < \epsilon_y < |x-y|$ for all $x \in F$ since F is closed, so that with the replacement $x - y = t$,

$$\int_F I(x)dx \leq \int_F\int_{\epsilon_y}^{\infty} \frac{2}{|t|^2}dtdy = \int_{F^C}\frac{2\delta(y)}{\epsilon_y}dy \leq \int_{F^C} 2dy = 2m(F^C) < \infty.$$

Whence, $I(x)$ is integrable so that $I(x) < \infty$ almost everywhere.

7. Suppose $f(x)$ is measurable on $\mathbb{R}^d$ so that $F(x,y) = y - f(x)$ is also measurable (being the linear combination of measurable functions) on $\mathbb{R}^{d+1}$. Then since $\{0\}$ is a Borel set, it is measurable and thus $F^{-1}(0) = \{(x,y) \in \mathbb{R}^{d+1} : y = f(x)\} = \Gamma$ is measurable. Furthermore, by a corrolary to Fubini's theorem, $\Gamma^x = \{y \in \mathbb{R} : y = f(x)\} = \{f(x)\}$ is a measurable function of x, and

$$m(\Gamma) = \int_{\mathbb{R}^d} m(\Gamma^x)dx = \int_{\mathbb{R}^d} m(\{f(x)\})dx = \int_{\mathbb{R}^d} 0dx = 0$$

as desired.

9. Since $f(x)$ is integrable on $\mathbb{R}^d$, it is measurable and the set $G = \{(x,y) \in \mathbb{R}^{d+1} : 0 \leq y \leq f(x)\}$ is measurable. Furthermore, $G^\alpha = \{x \in \mathbb{R}^d : 0 \leq \alpha \leq f(x)\} \supset E_\alpha$ and $G^\alpha \times [0,\alpha] \subset G$. Therefore,

$$m(G^\alpha \times [0,\alpha]) \leq \alpha m(E_\alpha) \leq m(G) = \int f$$

by Corollary 3.8. However, this is exactly,

$$m(E_\alpha) \leq \frac{1}{\alpha}\int f$$

as desired.

17.a) For fixed $x \in \mathbb{R}$ let $n = \lfloor x \rfloor$ then,

$$\int_{\mathbb{R}} |f_x(y)|dy = \int_n^{n+1} |a_n|dx + \int_{n+1}^{n+2} |-a_n|dx = 2|a_n| < \infty$$

so that $f_x(y)$ is integrable for all x. Now for fixed $y \in \mathbb{R}$, let $n = \lfloor y \rfloor$ thus

$$\int_{\mathbb{R}} |f^y(x)|dx = \int_n^{n+1} |a_n|dx + \int_{n-1}^{n+1} |-a_{n-1}|dx = |a_n| + |a_{n-1}| < \infty$$

where $a_{-1} \equiv 0$, whence $f_y(x)$ is also integrable for all y. Furthermore,

$$\int_{\mathbb{R}} f_x(y)dy = \int_n^{n+1} a_n dx + \int_{n+1}^{n+2} -a_n dx = 0$$

so that,

$$\int\int f(x,y)dydx = \int\int f_x(y)dydx = \int 0dx = 0.$$

b) Let $0 \leq y < 1$ so that $f^y(x) = a_0$ for $x \in [0,1)$ and 0 otherwise, whence,

$$\int_{\mathbb{R}} f^y(x)dx = \int_0^1 a_0 dx = a_0$$

and if $n \leq y < n+1$ with $n > 0$ we have,

$$\int_{\mathbb{R}} f^y(x)dx = \int_n^{n+1} a_n dx + \int_{n-1}^{n+1} -a_{n-1}dx = a_n - a_{n-1} = b_n.$$

It follows that $\int f^y(x)dy$ is integrable and

$$\int_0^\infty \int_0^\infty f^y(x)dxdy = \int_0^1 \int_0^\infty f^y(x)dxdy + \int_1^2 \int_0^\infty f^y(x)dxdy + ... = \int_0^1 a_0 dx + \int_1^2 b_1 dx + ... = b_0 + b_1 + ... = s.$$

c) If $s = 0$ then $f(x,y) = 0$ for all (x,y) and $\int_{\mathbb{R}^2} |f(x,y)|dxdy = 0$. Suppose then that $s > 0$, we have for any square $S_n = [n, n+1) \times [n, n+1)$ and any $(x,y) \in S_n$ we have $f(x,y) = a_n$ and,

$$\int_{S_n} |f(x,y)|dxdy = |a_n|$$

but,

$$\int_{\mathbb{R}^2} |f(x,y)|dxdy \geq \int_{\bigcup S_n} |f(x,y)|dxdy = \sum_{n=0}^\infty |a_n| \to \infty$$

because $|a_n| \not\to 0$.

21. a) By Proposition 3.9 and Corollary 3.7, $f(x-y)$ is a measurable function in $\mathbb{R}^{2d}$ since $f(x)$ is measurable, and $h(x,y) = g(y)$ is a measurable function in $\mathbb{R}^{2d}$ whence their product, $f(x-y)h(x,y) = f(x-y)g(y)$ is measurable in $\mathbb{R}^{2d}$.

b) By using Tonelli's theorem,

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|dxdy = \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)|dxdy = \|f\|_1 \int_{\mathbb{R}^d} |g(y)|dy = \|f\|_1\|g\|_1 < \infty$$

so that $f(x-y)g(y)$ is integrable.

c) Applying b) and Tonelli's theorem,

$$\int_{\mathbb{R}^d} |(f*g)(x)|dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y)dy \right| dx \leq \int_{\mathbb{R}^{2d}} |f(x-y)g(y)|dxdy = \|f\|_1\|g\|_1$$

so that $(f*g)(x)$ is $< \infty$ for almost every x, whence it is integrable a.e. x.

d) From c), it is obvious that if f and g are integrable, then $(f*g)(x)$ is as well. Furthermore, the left side of the equation in c) is exactly $\|(f*g)(x)\|_1$ so that we have $\|(f*g)(x)\|_1 \leq \|f\|_1\|g\|_1$. Again by inspection of c), we would have equality if $|\int f(x-y)g(y)dy| = \int |f(x-y)g(y)|dy$ which happens if $f(x-y)g(y) \geq 0$ for all y.

e) To show that $\hat{f}(\lambda)$ is bounded,

$$\hat{f}(\lambda) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\lambda}dx \leq \int_{\mathbb{R}^d} |f(x)|dx = \|f\|_1$$

since f is integrable, $\hat{f}(\lambda)$ is thus bounded. For continuity,

$$\begin{aligned} |\hat{f}(\lambda+h) - \hat{f}(\lambda)| &\leq \int_{\mathbb{R}^d} |f(x)||e^{-2\pi i x}(e^{-2\pi i h} - 1)|dx \\ &= |e^{-2\pi i h} - 1| \int_{\mathbb{R}^d} |f(x)|dx = \|f\|_1 |e^{-2\pi i h} - 1| \to 0 \end{aligned}$$

with $h \to 0$, hence $\hat{f}(\lambda)$ is a continuous function of λ. Finally,

$$\widehat{(f*g)}(\lambda) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y)g(y)dy \right) e^{-2\pi i x\lambda}dx$$

since $(f*g)(x)$ is integrable by d), so is $\widehat{(f*g)}$ so we may apply Fubini's theorem and letting $t = x - y$ get,

$$\widehat{(f*g)}(\lambda) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)e^{-2\pi i x\lambda}dxdy = \int_{\mathbb{R}^d} g(y)e^{-2\pi i y\lambda} \int_{\mathbb{R}^d} f(t)e^{-2\pi i t\lambda}dtdy = \hat{f}(\lambda)\hat{g}(\lambda).$$