Math 354
Assignment 6
Professor Jakobson

Eric Kissel
260477928

December 3, 2012

Problem 2. It is true that the lebesgue integral is relatively invariant under dialiation so that given $f(x)$ is integrable so too is $f(\delta x)$. It is also true that $\delta^d \int f(\delta x) = \int f(x)$. Now if we consider the limit as $\delta \to 1$, we obtain $lim\delta^d \int f(\delta x) = lim \int f(\delta x) = \int f(x)$ so that $||f(\delta x) - f(x)|| = 0$ as required.

Problem 5 a) To save tedious repetition, it is to be understood $z, w \in F$. $|\delta(x) - \delta(y)| = |\text{inf}|x - z| - \text{inf}|y - w|| \leq |\text{inf}|x - z - y + w|| \leq \text{inf}(|x - y| + |w - z|) \leq inf(|x - y|) = |x - y|$ as required.

b) Note that if $y \in F$, $\int_F f(x) = 0$ so $I(x) = \int_{F^c} \frac{\delta(x)}{|x-y|^2}$. Now since F is closed the complement is open, there exists $\epsilon_0 := 4\epsilon$ such that $(x - \epsilon_0, x + \epsilon_0) \in F^c$. Consider the interval $A = [x, x + \epsilon]$. $\delta(y) > 3\epsilon > \epsilon > |x - y|$ thus $\int_A \frac{1}{|x-y|} < I(x)$. Also, $f(x) \geq 0$ so from the properties of non-negative integrable functions, $\int_A f(x) \leq \int_{F^c} f(x)$. A is a closed interval, and $f(x)$ is riemann integral, and so $\int^L f(x) = \int^R f(x)$. Consider $lim_{p \to x} \int_p^{x+\epsilon} \frac{1}{|x-y|} dy = |ln|x - y|| = lim_{p \to x} |ln(\frac{\epsilon}{x-p})| = \ln \infty = \infty$. Since $I(x)$ is greater than that quantity, it is clear $I(x) = \infty$.

c) $I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} dy$. Note again that $\delta(y) = 0 \forall y \in F$ so $I(x) = \int_{F^c} \frac{\delta(y)}{|x-y|^2} dy$. Note too that $|x - y| > \delta(y)$ from the definiton of δ and since $x \in F$. Thus $I(x) \leq \int_{F^c} \frac{1}{|x-y|}$. Suppose x is not on the boundary of F, then $\frac{1}{|x-y|}$ is bounded by some M, and so is the measure of F^c so then the integral is clearly bounded by $M * m(F^c)$. Simply noticing the set of $\{x\}$ which are boundary points is a set of measure zero, we conlude that $I(x) < \infty$ for a.e. x.

(Problem 7) f is a measurable function and so $g := f^+ - f^-$ is also measurable and $g(x)$ is strictly non-negative and $m(f) \leq m(g)$. Consider two sets $E_1 = \{x : x \in \mathbb{R}\}$ and $E_2 = \{(x, y) : y = f(x)\}$. Since $E_1 = \mathbb{R}$, it follows E_1 is measurable, and since $f(x)$ is a measurable function, it follows $E = \{(x, y) : x \in E_1, (x, y) \in E_2\} = \Gamma$, $E \in \mathbb{R}^{d+1}$ is measurable by proposition 3.6. Further more, $m(E) = m(E_1)m(E_2)$. Now $m(E_2) = m(\{(x, y) : y = g(x)\})$. We use theorem 3.2 to obtain $m(E_2) = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \chi_{(x,y)} dy dx$. Note that for each x the measure of y is zero; it is simply a point, so we eventually obtain $m(E_2) = 0$. Finally, by proposition 3.6, $m(E) = m(E_1)m(E_2) = 0$ as required.

(Problem 9) $m(E_\alpha) = \int \chi_{E_\alpha}$. Also if $\chi_{E_\alpha} = 1$, $\frac{f(x)}{\alpha} > 1 = \chi_{E_\alpha}$ and if $\chi_{E_\alpha} = 0$, $f(x) \geq 0 = \chi_{E_\alpha}$

thus, $\chi_{E_\alpha} \leq \frac{f(x)}{\alpha} \forall x$ so by monotonicity of the integral, $m(E_\alpha) = \int \chi_{E_\alpha} \leq \int \frac{f(x)}{\alpha} = \frac{1}{\alpha}\int f(x)$ as required.

(Problem 17) a) Since both functions are simple functions with a finite sum (up to s) they must both be integrable. Suppose we fix x, then there exists a single integer n such that $n \leq x \leq n+1$. Notice that we have a simple function, and the for all y such that $y < n, y > n+2$, $f(x,y) = 0$ so $\int f_x(y)dy = \int_{[n,n+1]} f_x(y)dy + \int_{[n+1,n+2]} f_x(y)dy = 1 * a_n - 1 * a_n = 0$ so $\int(\int f(x,y)dy)dx = 0$

b) Fix y. Note that for the interval $0 \leq y \leq 1$ occurs for $n = 0$, $n \leq y \leq n+1$, and would occur for $n = -1$, $n+1 \leq y \leq n+1$, but $n \geq 0$. Thus for the first interval, we have $\int f^y(x)dx = a_0$. For subsequent intervals, the interval occurs for $n+1 \leq y \leq n+2$ at an n value one lower than the same interval will occur for $n \leq y \leq n+1$. Thus for fixed y, $y > 1$, $\int_{[n,n+1]} f^y(x)dx = a_n - a_{n-1}$. Now we integrate over all $y \geq 0$ since it is zero otherwise. $g(y) = \int f^y(x)dx$ is a simple function, so its integral is given by $\sum_{i=0}^{\infty} a_n - a_{n-1} + a_0 = a_n = s$ as required.

c) Notice that our simple function $g(y) = a_k + a_{k-1}$ so for sufficient large n our integral is larger than $\sum_{i=k}^{\infty} s - \epsilon$ which fails the nth term test and diverges.

(Problem 21) a) By proposition 3.9, $f(x-y)$ is measurable on $\mathbb{R}^{2d}$ and the product of measurable functions is also measurable, so we conclude $f(x-y)g(y)$ is a measurable function.

b) By theorem 3.2, since our function is measurable, $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y) = \int_{\mathbb{R}^{2d}} f(x-y)g(y)$ but both of these are integrable, and we may integrate first $f(x-y)$ and, after noting the translation invariance of the obtained result, integrate again to obtain $\int f(x-y)g(y) = \int f(x)\int g(y) \leq \infty$.

c) By theorem 3.2, since our function is measurable, $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y) = \int_{\mathbb{R}^{2d}} f(x-y)g(y)$ but both of these are integrable, and we may integrate first $f(x-y)$ and, after noting the translation invariance of the obtained result, integrate again to obtain $\int f(x-y)g(y) = \int f(x)\int g(y) \leq \infty$. (I don't know how to do this one).

d) $||f * g||_{L(\mathbb{R}^d)} = \int|\int f(x-y)g(y)dx|dy \leq \int\int|f(x-y)||g(y)|dxdy = \int||f(x-y)||_{L(\mathbb{R}^d)}|g(y)|$ but the integral is translation invariant, so we have $\int||f(x-y)||_{L(\mathbb{R}^d)}g(y) = ||f||_{L(\mathbb{R}^d)} * ||g||_{L(\mathbb{R}^d)}$. Finally, notice that if all functions are positive then the aboslute values become unnecessary and we obtain $||f * g||_{L(\mathbb{R}^d)} = ||f||_{L(\mathbb{R}^d)}||g||_{L(\mathbb{R}^d)}$

Problem 3. Given a sequence of functions $f_k(x)$ converge to $f(x)$, by egarov there exists a set A such that if $x \notin A$, $||f_k(x) - f(x)|| < \epsilon$ and $m(A) < \epsilon$. It is true then that $E = \{x : ||f_k(x) - f(x)|| > \epsilon\} = A$. But by definition $m(A) < \epsilon$, and so $(\rightarrow)$ is complete. The converse is true. This is because the limiting function and f differ at most by a set of measure 0 in L_1, and so their integrals must be identical.