5.

(a) Let $x, y \in \mathbb{R}$, $\varepsilon > 0$. Then, there is a $z \in F$ such that $|y - z| \leqslant \delta(y) + \varepsilon$. Then:

$$\delta(x) \leqslant |x - z| \leqslant |x - y| + |y - z| \leqslant |x - y| + \delta(y) + \varepsilon$$

So $\delta(x) - \delta(y) \leqslant |x - y| + \varepsilon$ for any $\varepsilon > 0$, so $\delta(x) - \delta(y) \leqslant |x - y|$. Symmetrically, we obtain that $|\delta(x) - \delta(y)| \leqslant |x - y|$, and δ satisfies the described Lipschitz condition, so δ is continuous. □

(b) Suppose $x \notin F$.

Then I claim that $\delta(x) > 0$. Indeed, since $\delta(x) = \inf\{|x - y| : y \in F\}$, then if $\delta(x) = 0$, we would obtain a sequence of elements of F converging to x. Since F is closed, then $x \in F$, which is a contradiction. So $\delta(x) > 0$. Since δ is continuous, then there is an interval $[x - \varepsilon, x + \varepsilon]$ such that $\delta(x) \geqslant M$ on this interval. So:

$$I(x) = \int \frac{\delta(y)}{|x - y|^2} dy \geqslant \int_{[x-\varepsilon, x+\varepsilon]} \frac{M}{|x - y|^2} dy \geqslant M \int_x^{x+\varepsilon} \frac{1}{|x - y|^2} dy = \infty$$

□

(c) First, $\delta(x) = 0$ iff $x \in F$, from above. In addition, $\delta(y) \leqslant |x - y|$ whenever $x \in F$ and $y \in F^c$. By Fubini, we have:

$$\begin{aligned}
\int_F I(x) dx &= \int_F \left(\int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2} dy \right) dx = \int_{\mathbb{R}} \delta(y) \left(\int_F \frac{1}{|x - y|^2} dx \right) dy \\
&= \int_{F^c} \delta(y) \left(\int_{|x-y| \geqslant \delta(y)} \frac{1}{|x - y|^2} dx \right) dy \\
&= \int_{F^c} \delta(y) \left(\int_{-\infty}^{y-\delta(y)} \frac{1}{|x - y|^2} dx + \int_{\delta(y)+y}^{\infty} \frac{1}{|x - y|^2} dx \right) dy \\
&= \int_{F^c} \delta(y) \frac{2}{\delta(y)} dy = 2m(F^c) < \infty
\end{aligned}$$

Since this integral is finite, then $I(x) < \infty$ for a.e. $x \in F$. □

6.

(a) Let f be defined as follows, where $n \geqslant 2$:

$$f(x) = \begin{cases} n^4 x + n - n^5 & \text{if } x \in [n - \frac{1}{n^3}, n); \\ n & \text{if } x \in [n, n + \frac{1}{n^3}); \\ -n^4 x + 2n + n^5 & \text{if } x \in [n + \frac{1}{n^3}, n + \frac{2}{n^3}); \\ 0 & \text{otherwise.} \end{cases}$$

Such a function is indeed positive, continuous and can be drawn as a sequence of trapezoids of height n and bases of length $\frac{1}{n^3}$ and $\frac{3}{n^3}$ for $n \geqslant 2$. Then:

$$\int f dx = \sum_{n=2}^{\infty} n \frac{\frac{1}{n^3} + \frac{3}{n^3}}{2} = 2 \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

So f is integrable. In addition, $\limsup_{x\to\infty} f = \infty$ since for any N, $\sup_{x\geqslant N} f(x) \geqslant N$

(b)

7.

Note that $\Gamma^c = U \cup V$, where $U = \{(x,y) \in \mathbb{R}^d : y < f(x)\}$ and $V = \{(x,y) \in \mathbb{R}^d : y > f(x)\}$. Then:

$$U = \bigcup_{a \in \mathbb{Q}} f^{-1}(-\infty, a) \times (-\infty, a)$$
$$V = \bigcup_{a \in \mathbb{Q}} f^{-1}(-\infty, a) \times (a, \infty)$$

since $\mathbb{Q}$ is dense in $\mathbb{R}$. Since f is a measurable function, then U and V are measurable sets, so Γ^c is measurable and Γ is measurable.

Since Γ is measurable, then χ_Γ is measurable and non-negative. By Tonelli:

$$m(\Gamma) = \int_{\mathbb{R}^{d+1}} \chi_\Gamma dy dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \chi_\Gamma(x,y) dy \right) dx$$

But for a fixed x, we have:

$$\int_{\mathbb{R}} \chi_\Gamma(x,y) dy = m(\{y : (x,y) \in \Gamma\}) = m(\{f(x)\}) = 0$$

So indeed, $m(\Gamma) = 0$. □

9.

Note the following:

$$\alpha m(E_\alpha) = \int \alpha \chi_{E_\alpha} dx$$

Since $f(x) > \alpha$ on E_α, then:

$$m(E_\alpha) = \frac{1}{\alpha} \int \alpha \chi_{E_\alpha} dx \leqslant \frac{1}{\alpha} \int f dx$$

□

14.

(a) The unit ball in $\mathbb{R}^2$ consists of all points with $(x^2 + y^2)^{1/2} \leqslant 1$, i.e. all points satisfying $y \leqslant (1 - x^2)^{1/2}$ or $y \geqslant -(1 - x^2)^{1/2}$. By Corollary 3.8, we have:

$$\begin{aligned}
v_2 &= m(B_1) \\
&= m(\{(x,y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant (1 - x^2)^{1/2}\}) + m(\{(x,y) \in \mathbb{R}^2 : -(1 - x^2)^{1/2} \leqslant y \leqslant 0\}) \\
&= 2m(\{(x,y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant (1 - x^2)^{1/2}\}) = 2\int_{-1}^{1} (1 - x^2)^{1/2} dx \\
&= 0 + \frac{\arcsin(1) - \arcsin(-1)}{2} = \pi
\end{aligned}$$

□

(b)

(c)

15.

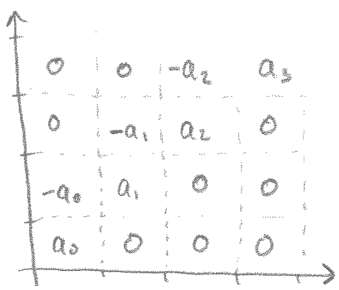
Let $s_k(x) = \sum_{n=1}^{k} 2^{-n} f(x - r_n)$. Since f is measurable, then $2^{-n} f(x - r_n)$ is measurable, so $s_k(x)$ is measurable. Then, by Corollary 1.10:

$$\int F dx = \int \sum_{n=1}^{\infty} 2^{-n} f(x - r_n) dx = \sum_{n=1}^{\infty} \int 2^{-n} f(x) dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2$$

so F is integrable. If the series describing F diverged on a set of positive measure, then $\int F dx$ would be infinite, so it must be that the series describing F converges for a.e. $x \in \mathbb{R}$.

17.

(a) Drawing the values of f in the plane, some results become evident:



Fixing $y \in [0, 1)$, we have that:

$$\int |f^y| dx = a_0 = b_0 \leqslant s < \infty$$

if $y \in [n, n+1)$ for $n \geqslant 1$, then:

$$\int |f^y| dx = a_n + a_{n-1} = \leqslant 2s < \infty$$

Similarly, fixing $x \in [n, n+1)$, we have that:

$$\int |f_x| dy = a_n + a_n \leqslant 2s < \infty$$

So that f_x and f^y are integrable. Indeed, for all x:

$$\int f_x dy = a_n - a_n = 0$$

So that:

$$\int \left(\int f(x, y) dy \right) dx = 0$$

□

(b) Fix $y \in [0,1)$, we see that:

$$\int f^y dx = a_0$$

If $y \in [n, n+1)$, with $n \geqslant 1$, then:

$$\int f^y dx = a_n - a_{n-1}$$

Thus:

$$\int \left(\int f(x,y) dx \right) dy = a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) = \lim_{n\to\infty} a_n = s$$

□

(c) We have that:

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)| dxdy = 2 \sum_{n=0}^{\infty} a_n$$

Since $\{b_k\}$ is a positive sequence, then for any N, the tail of this series has $\sum_{n\geqslant N} a_n \geqslant s$. Since the tail cannot be made arbitrarily small, then the series diverges, so:

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)| dxdy = \infty$$

□

21.

(a) By Proposition 3.9, we know that the function $\tilde{f}(x,y) = f(x-y)$ is measurable on $\mathbb{R}^{2d}$. In addition, by Corollary 3.7, $\tilde{g}(x,y) = g(y)$ is measurable on $\mathbb{R}^{2d}$. Since the product of measurable function is measurable, then $\tilde{f}(x,y)\tilde{g}(x,y) = f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$. □

(b) By Tonelli's theorem, we have:

$$\int \int |f(x-y)g(y)| dydx = \int |g(y)| \int |f(x-y)| dxdy = \left(\int |f| \right) \left(\int |g| \right) < \infty$$

So $f(x-y)g(y)$ is integrable. □

(c) By Fubini's theorem, since $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$, then the slice f_x is integrable on $\mathbb{R}^d$ for a.e. x, so $f * g$ is well-defined for a.e. x. □

(d) If f, g are integrable, then:

$$\int \left| \int f(x-y)g(y) dy \right| dx \leqslant \int \int |f(x-y)g(y)| dydx < \infty$$

As shown in (b), we indeed have:

$$\|f*g\| = \int \left| \int f(x-y)g(y)dy \right| dx \leqslant \int \int |f(x-y)g(y)| dy dx = \|f\| \|g\|$$

with equality holding if $|\int f(x-y)g(y)dy| = \int |f(x-y)g(y)|dy$, in particular when f and g are non-negative. □

(e) We first show that $\hat{f}$ is bounded:

$$|\hat{f}(\xi)| \leqslant \int |f(x)||e^{-2\pi i x\cdot\xi}| dx \leqslant \int |f| < \infty$$

So it is bounded. Now, let $n \geqslant 1$. Then:

$$\begin{aligned}
|\hat{f}(\xi + \frac{1}{n}) - \hat{f}(\xi)| &= \left| \int f(x)(e^{-2\pi i x\cdot(\xi+\frac{1}{n})} - e^{-2\pi i x\cdot\xi}) dx \right| \\
&= \left| \int f(x) e^{-2\pi i x\cdot\xi} (e^{-2\pi i x\cdot\frac{1}{n}} - 1) dx \right| \\
&\leqslant \int |f(x) e^{-2\pi i x\cdot\frac{1}{n}} - f(x)| dx
\end{aligned}$$

Since $f(x)e^{-2\pi i x\cdot\frac{1}{n}} \to f(x)$ for a.e. x as $n \to \infty$, and $|f(x)e^{-2\pi i x\cdot\frac{1}{n}}| \leqslant |f(x)|$, which is integrable, then by the Dominated Convergence Theorem, we get that:

$$|\hat{f}(\xi + \frac{1}{n}) - \hat{f}(\xi)| \leqslant \int |f(x) e^{-2\pi i x\cdot\frac{1}{n}} - f(x)| dx \to 0$$

So $\hat{f}$ is continuous. In addition:

$$\widehat{(f*g)}(\xi) = \int \left(\int f(x-y)g(y)dy \right) e^{-2\pi i x\cdot\xi} dx$$

By Fubini:

$$\begin{aligned}
\widehat{(f*g)}(\xi) &= \int g(y) \left(\int f(x-y) e^{-2\pi i x\cdot\xi} dx \right) dy \\
&= \int g(y) e^{-2\pi i y\cdot\xi} \left(\int f(x-y) e^{-2\pi i (x-y)\cdot\xi} dx \right) dy \\
&= \int g(y) e^{-2\pi i y\cdot\xi} \left(\int f(x) e^{-2\pi i x\cdot\xi} dx \right) dy \\
&= \left(\int g(x) e^{-2\pi i x\cdot\xi} dx \right) \left(\int f(x) e^{-2\pi i x\cdot\xi} dx \right) = \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

□

22.

Note that:

$$\frac{1}{2}\int [f(x) - f(x-\xi')]e^{-2\pi i x\cdot\xi}dx = \frac{1}{2}\hat{f}(\xi) - \frac{1}{2}\int f(x)e^{-2\pi i\left(x+\frac{\xi}{2|\xi|^2}\right)\cdot\xi}dx$$
$$= \frac{1}{2}\hat{f}(\xi) - \frac{1}{2}\int f(x)e^{-2\pi i x\cdot\xi}e^{\pi i}dx = \hat{f}(\xi)$$

Suppose $|\xi| \to \infty$. then $|\xi'| = \frac{1}{2|\xi|} \to 0$. By Proposition 2.5, we have:

$$\hat{f}(\xi) = \frac{1}{2}\int [f(x) - f(x-\xi')]e^{-2\pi i x\cdot\xi}dx \leqslant \frac{1}{2}\int |f(x) - f(x-\xi')|dx \to 0$$

□