5. Suppose F is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x)$ denote the distance from x to F, that is,

$$\delta(x) = d(x, F) = \inf\{|x - y| : y \in F\}.$$

Consider

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2} dy.$$

(a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition

$$|\delta(x) - \delta(y)| \leq |x - y|.$$

Solution

This was proven in Assignment 1, nevertheless here is the proof for completeness.

Proof Let $\varepsilon > 0$ be given. The definition of infimum implies that for $\varepsilon > 0, \exists a_\varepsilon \in F$ such that

$$\Big(\forall\, \varepsilon > 0\Big)\Big(\exists a_\varepsilon \in F\Big)\Big(\inf_{a \in F} d(x, F) + \varepsilon > d(x, a_0)\Big). \tag{1}$$

By the triangle inequality ($d(x, y) := |x - y|$ being a distance), we have for $a_1, a_2 \in F$ satisfying (1) for the given ε and for the points y, x respectively that

$$d(x, F) \leq d(x, a_1) \leq d(x, y) + d(y, a_1) \leq d(x, y) + d(y, F) + \varepsilon$$

and similarly

$$d(y, F) \leq d(y, a_2) \leq d(x, y) + d(x, a_2) \leq d(x, y) + d(x, F) + \varepsilon.$$

Combining the two statements, we find that

$$|d(x, F) - d(y, F)| \leq d(x, y) + \varepsilon$$

where ε is arbitrary. Now, it is clear that from the definition of continuity, one can take $\delta \equiv \varepsilon$ satisfies the requirements and so we conclude that for fixed F, $d(x, F)$ is a continuous function satisfying the Lipschitz condition, as the above condition can be rewritten for the particular given metric as

$$|\delta(x) - \delta(y)| \leq |x - y|.$$

■

(b) Show that $I(x) = \infty$ for each $x \notin F$.

Solution
We get from the Lipschitz condition that $\exists\varepsilon > 0$ such that $0 \leq 2\varepsilon \leq \delta(y)$

$$I(x) \geq \int_{x-\varepsilon}^{x+\varepsilon} \frac{\varepsilon dy}{(x-y)^2} = \varepsilon \int_{-\varepsilon}^{\varepsilon} \frac{dy_*}{y_*^2} = \infty$$

taking the integral over a smaller set, and using the translation invariance of the Lebesgue measure (that is making a change of variable $(x-y) = y_*$). Since on the smaller interval the integral diverges, then $I(x) = \infty$ if $x \notin F$.

(c) Show that $I(x) < \infty$ for a.e. $x \in F$.

Solution
Using the fact that $\delta(y) = 0$ for each $y \in F$, we want to show that

$$\begin{aligned}\int_F \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} dy &= \int_F \int_{F^{\complement}} \frac{\delta(y)}{|x-y|^2} dy dx \\ &= \int_{F^{\complement}} \delta(y) \left(\int_F (x-y)^{-2} dx \right) dy\end{aligned}$$

using Fubini in the last step. Now, fix $y \in F^{\complement}$ and let $x \in F$; in such case, we know that $|x-y| \geq \delta(y)$ and thus we also have the inclusion $F \subset D := \{x \in \mathbb{R} : |x-y| \geq \delta(y)\}$. We can thus enlarge the integral by taking the larger set D and so

$$\int_{F^{\complement}} \delta(y) \left(\int_F (x-y)^{-2} dx \right) dy \leq \int_{F^{\complement}} \delta(y) \left(\int_D x^{-2} dx \right) dy$$

and we can now invoke Stein and Shakarchi, p.63, who show that for the function $f(x) = x^{-d-1}$, (in our case $(x-y)^{-2}$, which is nowhere zero since we assume $x \in F, y \in F^{\complement}$), we get the bound for

$$\int_{|x-y| \geq \delta(y)} f(x) dx \leq \frac{2}{\delta(y)}$$

using the relative dilation-invariance and translation invariance of the Lebesgue measure. The justification is explicited in the book, which I reproduce below for convenience, consisting at using the compactness of the set $F^{\complement}$, decomposed into sets

$$A_k = \{x \in \mathbb{R} : 2^k \delta(y) < |x-y| \leq 2^{k+1}\delta(y)\}$$

for y fixed and getting an approximation from above by a simple function $g(x) = \sum_{k=0}^{\infty} (2^k \delta(y))^{-2} \chi_{A_k}(x)$. The sets A_k are obtained from dilation of the sets $\mathcal{A} = \{1 \leq |x-y| < 2\}$. From there,

$$\int g = \sum_{k=0}^{\infty} \frac{m(A_k)}{(2^k \delta(y))^2} = m(\mathcal{A}) \sum_{k=0}^{\infty} \frac{2^k \delta(y)}{(2^k \delta(y))^2}$$

Coming back to our case, we get

$$\int_{F^{\complement}} \int_F \frac{1}{(x-y)^2} dx dy \leq \int_{F^{\complement}} 2 \frac{\delta(y)}{\delta(y)} = 2 \int_{\mathbb{R}} \chi_{F^{\complement}} = 2m(F^{\complement}) < \infty$$

by assumption.

This work and the question are related on extensions of lemmas by Marcinkiewicz and Fine, which was the work of Stein and Ostrow (1957).

7. Let $\Gamma \subset \mathbb{R}^d \times \mathbb{R}, \Gamma = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$, and assume f is measurable on $\mathbb{R}^d$. Show that Γ is a measurable subset of $\mathbb{R}^{d+1}$, and $m(\Gamma) = 0$.

Solution

This resembles question 37 in the previous assignment, except for the notable exception that now f is measurable rather than being continuous. Consider an interval of unit length, of the form $y \in \mathbb{R} : y \in E_n := [n, n+1)$ and wlog of generality suppose that f is positive (otherwise, consider a partition of $f = f^+ - f^-$ and cover the space accordingly) and finite-valued. Let $\varepsilon > 0$ be given. By Lusin's theorem, we can find closed set $F_\varepsilon^{(n)}$ such that $F_\varepsilon^{(n)} \subset E$ and $m(E - F_\varepsilon^{(n)}) \leq \varepsilon/2$. such that $f|_{F_\varepsilon^{(n)}}$ is continuous. Using then uniform continuity on the closed and bounded interval, then we can tile the unit interval with N different hypercubes of length $\varepsilon = 2^{-k}$,. By absolute continuity, we have that there exists δ_ε for each given ε which corresponds here to the base of each hypercube. Taking $\delta = \min\{1/2^d N, \delta_\varepsilon\}$ given, then the total hypervolume of the cover for $\Gamma|_{[n,n+1]}$ is $N \times 2^d\delta/2^k \leq 2/2^k$. This holds for all k, thus from this we infer that $m(\Gamma|_{F_\varepsilon^{(n)}}) = 0$, taking $k \to \infty$. Since our choice of partition of the real line is countable, we can then using countable subadditivity to take

$$\begin{aligned}
m(\Gamma) &= m\left(\bigcup_{n\in\mathbb{Z}} \Gamma \cap F_\varepsilon^{(n)}\right) + m(\Gamma \cap (E_n \setminus F_\varepsilon^{(n)})) \\
&\leq \sum_{n\in\mathbb{Z}} m(\Gamma \cap F_\varepsilon^{(n)}) + \sum_{n\in\mathbb{Z}} m(\Gamma \cap (E_n \setminus F_\varepsilon^{(n)})) \\
&= 0
\end{aligned}$$

since each individual $\Gamma \cap F_\varepsilon^{(n)}$ has measure zero for $n \in \mathbb{Z}$. Indeed, the sets $E_n \setminus F_\varepsilon^n$ has measure 2^{-k-1} and we can infer that in each interval the measure is zero since ε can be chosen arbitrarily small.

$$\max_{x\in E_n\setminus F_\varepsilon^{(n)}} f(x)m(E_n \setminus F_\varepsilon^{(n)}) + 1/2^{k-1} = \max_{x\in E_n\setminus F_\varepsilon^{(n)}} f(x)\frac{1}{2^k} + \frac{1}{2^{k-1}} \xrightarrow{k\to\infty} 0.$$

Therefore, $m(\Gamma) = 0$. The measurability of Γ in $\mathbb{R}^{d+1}$ is immediate from Corollary 3.8; since f is measurable on $\mathbb{R}^d$ if and only if Γ is measurable in $\mathbb{R}^d \times \mathbb{R}$ and having the former gives us the desired result. Indeed, we could construct one set with $f(x) = g(x) - g^*(x)$ where the function we consider $\mathcal{A} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}$ and $\mathcal{A}^* = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y < f(x)\}$. Using the measurability, we would then get by linearity that

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} [g(x) - g^*(x)]dx = m(\mathcal{A}) - m(\mathcal{A}^*) = 0.$$

9. **Tchebychev inequality.** Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha}\int f.$$

Proof Recall that $m(E_\alpha) = \int \chi_{E_\alpha}$. Since $f(x) > \alpha$ on E_α, we have

$$\int f \leq \int_{E_\alpha} f \leq \alpha \int_{E_\alpha} = \alpha \int \chi_{E_\alpha} = \alpha m(E_\alpha).$$

■

15. Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F a.e. is unbounded in any interval.

Solution
Want to show that

$$F(x) = \sum_{n=1}^{\infty} \frac{2^{-n}}{\sqrt{|x - r_n|}} < \infty \text{ a.e. Lebesgue}$$

First, consider the sequence of function

$$g_n(x) = \begin{cases} x^{-1/2} & \text{if } x \in (1/n, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

agrees a.e.x with $f(x)$. Also, we can use the translation invariance of f to deduce the equality for any given translation by a rational. Given that $g\chi_{(1/n,1]}$, we get that the integral of f is equal to

$$\int_{(} 0, 1] f(x)dx = \int_{[1/n,1]} g_n(x)dx$$

since we do not care about sets of measure zero. Moreover, since $g_n(x)$ is monotonically increasing and positive, we can use Monotone convergence theorem to get

$$\lim_{n\to\infty} \int_{1/n}^{1} g_n(x)dx = \lim_{n\to\infty} 2\left(1 - \frac{1}{\sqrt{n}}\right) = 2$$

as the value of the integral. Using again Monotone convergence theorem in conjunction with Corollary 1.10, since the series $2^{-n} f(x - r_n)$ is measurable and positive for each n, we get

$$\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty}] \int a_k(x)dx$$

interchange of integral and summations. In our specific case, this translates to the following:

$$\int F dx = \int \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n) dx = \sum_{n=1}^{\infty} \frac{1}{2^n} \int f(x - r_n) dx = 2 \sum_{n=1}^{\infty} \frac{1}{2^n} = 2(2 - 1)$$

using the properties of geometric series. This shows that F is integrable and converges. However, in any interval, we can using the density of the rationals $\mathbb{Q}$ in the real find r_k such that $f(x - r_k) > N$ for any arbitrarily large K in the interval $A_k = (r_k - 2^{-k}, r_k + 2^{-k})$ and the series is unbounded on any such interval. Yet, the series converge, indeed, recall from the Borel-Cantelli lemma that if $\sum_{k=1}^{\infty} m(A_k) < \infty$, then $m\left(\bigcup_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k\right) = 0$. We have $m(A_n) = 2^{-n+1}$ and $\sum_{n=1}^{\infty} 2a_n < \infty$, where $a_n = 1/2^n$ is used for convenience, as the result holds more generally for convergent series. Therefore, $m(\limsup_{k\to\infty} A_k) = 0$ by Borel-Cantelli. In fact, the set $\{x : f(x) = \infty\} \subset \mathbb{Q} \cup \limsup_{k\to\infty} A_k := S$. The above holds if and only if $S^{\complement} \subset \{x : f(x) < \infty\}$; if $x \in S^{\complement}$, then $x \notin \mathbb{Q}$ **and** $x \notin \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$, so $\exists N$ such that $\forall\, n \geq N, x \notin \bigcup_{k=N}^{\infty}$. Fix this N, then $|x - r_k| \geq a_k = \frac{1}{2^k}$ and we get that

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{1}{2^n f(x - r_n)} \\ &= \sum_{n=1}^{N-1} \frac{1}{2^n} \frac{1}{\sqrt{x - r_n}} + \sum_{n=N}^{\infty} \frac{1}{2^n \sqrt{x - r_n}} \\ &\leq \sum_{n=N}^{\infty} \frac{1}{2^{n/2}} < 1 + \sqrt{2} < \infty \end{aligned}$$

as $\sum_{n=1}^{N-1} \frac{1}{2^n} \frac{1}{\sqrt{x-r_n}}$ is finite unless $x = r_m$ for some m, but this equality is true only on a set of measure zero (rationals are countable), entailing that $x \notin \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty}$ which in turn imply $f(x) < \infty$ so $x \notin \{x : f(x) = \infty\}$. Finally, by the contrapositive, the set $\{x : f(x) = \infty\} \subset \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$, which has measure zero.

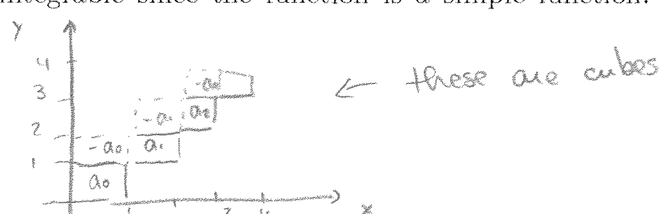
17. Suppose f is defined on $\mathbb{R}^2$ as follows: $f(x, y) = a_n$ if $n \leq x < n+1$ and $n \leq y < n+1, (n \geq 0)$; $f(x, y) = -a_n$ if $n \leq x < n + 1$ and $n + 1 \leq y < n + 2, (n \geq 0)$; while $f(x, y) = 0$ elsewhere. Here $a_n = \sum_{k \leq n} b_k$ with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

   (a) Verify that each slice f^y and f^x is integrable. Also for all x, $\int f_x(y)dy = 0$, and hence $\int(\int f(x, y)dy)dx = 0$.

   Solution

   A picture helps here.

   Clearly, each slice is integrable since the function is a simple function. For $x \in \mathbb{R}$



   in some interval $E_n = \{x : x \in [n, n + 1)\}$, we have $f^y(x) = -a_n\chi_{E_n} + a_{n+1}\chi_{E_{n+1}}$, which is integrable by the definition of Lebesgue integral, since the function is simple. We have for $y \in [0,1)$ that $f^y(x) = a_0$ (see picture). Then, for $f_x(y)$ we get a similar result, namely that if for $n \geq 0$, $A_n = \{y : y \in [n, n+1)\}$ and this time for any given x, $\int_x f(y) = a_n\chi_{A_n} - a_n\chi_{A_{n+1}}$ by definition since the measure of the cubes (or slices) are equal. This latter part is zero, since a_n and $-a_n$ cancel each other. Since $\int f_x(y)dy = a_n\chi_{A_n} - a_n\chi_{A_{n+1}} = 0$, then $\int(\int f_x(y)dy)dx = \int 0dx = 0$. On the other hand, since $a_n - a_{n-1} = b_n$, we get

$$\int\left(\int f^y(x)dx\right)dy = \lim_{n\to\infty}\sum_{i=0}^{n} b_n\chi_{E_n} = s$$

(b) However, $\int f^y(x)dx = a_0$ if $0 \leq y < 1$, and $\int f^y(x)dx = a_n - a_{n-1}$ if $n \leq y < n+1$ with $n \geq 1$. Hence $y \mapsto \int f^y(x)dx$ is integrable on $(0,\infty)$ and

$$\int\left(\int f(x,y)dx\right)dy = s.$$

This was explained above.

(c) Note that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \infty$.

Solution
The above explanation show that Fubini theorem does not apply, and so we must conclude that $\int\int|f(x,y)|dxdy = \int_{\mathbb{R}\times\mathbb{R}} 2\sum_{k=0}^{n} b_k = 2\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k = \infty$ as each term of the sequence $\{b_k\}$ appears infinitely often and the Lebesgue integral of a constant is not finite (and doesn't exist).

19. Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}}^{d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

Solution
Given $f \geq 0$ integrable, we can write

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty m\{x \in E_\alpha : f(x) > \alpha\}d\alpha$$

If E_α is measurable, we can write $\int_{\mathbb{R}} \chi_{E_\alpha}(\alpha)d\alpha = m(E_\alpha)$. Then, since $f(x) = \int_0^\infty \chi_{[0,f(x)]}(\alpha)d\alpha$ as a function of α, we get as $f \in L^1(\mathbb{R}^d)$ that Fubini-Tonelli theorem applies and so

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)dx &= \int_{\mathbb{R}^d}\int_0^\infty \chi_{[0,f(x)]}d\alpha dx \\
&= \int_0^\infty\left(\int_{\mathbb{R}} \chi_{[0,f(x)]}(\alpha)dx\right)d\alpha \\
&= \int_0^\infty m\{x : f(x) > \alpha\}dt \\
&= \int_0^\infty m(E_\alpha)d\alpha
\end{aligned}$$

since $\chi_{[0,f(x)]}(\alpha)$ is one only if $\alpha < f(x)$. This proves the result.

21. Suppose that f and g are measurable functions on $\mathbb{R}^d$.

(a) Prove that $f(x-y)g(y)$ are measurable on $\mathbb{R}^{2d}$.

Solution

From the definition of measurable functions in chapter 1, the function f is measurable on $E \subset \mathbb{R}^d$, if $\forall\, a \in \mathbb{R}$, the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable. First, we show that the functions $f(x), g(y)$ are measurable in $\mathbb{R}^{2d}$. By Corollary 3.7, we know that for f measurable on $\mathbb{R}^d$, then $\tilde{f}(x,y) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \tilde{f}(x,y) < a\}$ is measurable. We get similar result for $\tilde{g}(x,y) = g(y)$. Using translation invariance of the Lebesgue measure, we have that $f(x-y) = \tilde{f}(x,y)$ is measurable (see Proposition 3.9). Now, from chapter 1, by property 5, we have measurability of $\tilde{f}^2(x-y)$ and $\tilde{g}^2(y)$ and the product of functions $\tilde{f}\tilde{g}$ is also measurable, since one can write $\tilde{f}\tilde{g} = \frac{1}{4}[(\tilde{f}+\tilde{g})^2 - (\tilde{f}-\tilde{g})^2]$. These proposition follow by translation-invariance of the Lebesgue measure.

(b) Show that if f and g are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

Solution

Using integrability of f and g, we find as $f(x-y)g(y)$ is measurable that

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|dxdy &\le \int_{\mathbb{R}^d} \|f\|_{L^1}|g(y)|dy \\ &= \|f\|_{L^1}\|g\|_{L^1} < \infty.\end{aligned}$$

and so by Fubini theorem, is integrable on the product space. Note that we can also interchange limits. $(f * g) \in L^1(\mathbb{R}^{2d})$ and $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$

(c) Recall the definition of the convolution of f and g given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

Show that $f * g$ is well-defined for a.e. x (that is, $f(x-y)g(y)$ is integrable on $\mathbb{R}^d$ for a.e. x).

Solution

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)dydx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)dxdy \\ &= \int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} f(x)dxdy \\ &= \left(\int_{\mathbb{R}^d} f(x)dx\right)\left(\int_{\mathbb{R}^d} g(y)dy\right)\end{aligned}$$

using Fubini-Tonelli theorem, the translation invariance of the Lebesgue integral. Also,

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)dydx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(x-y)dydx \\ &= \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d} g(x-y)dx\right)dy\end{aligned}$$

(d) Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)},$$

with equality if f and g are non-negative.

Solution
Again, repetition of the above, this time with absolute values.

$$\begin{aligned}
\|f * g\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)| dy dx \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)| dx dy \\
&= \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)| dx dy \\
&= \int_{\mathbb{R}^d} |g(y)| \|f\|_{L^1(\mathbb{R}^d)} dy \\
&= \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} < \infty
\end{aligned}$$

where the first line is by definition, the second uses the triangle inequality for functions and Fubini theorem. Clearly, we have equality of $|f(x-y)g(y)| = |f(x-y)g(y)|$ if $f(), g()$ are a.e. positive.

(e) The Fourier transform of an integrable function f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Check that $\hat{f}$ is bounded and is a continuous function of ξ. Prove that each ξ one has

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Solution
Denote for simplicity $(f * g)(\xi) \equiv h(\xi)$, then

$$\begin{aligned}
\hat{h}(\xi) = \int_{\mathbb{R}^d} h(x) e^{-2\pi i \xi x} dx \\
&= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(x-y) g(y) dy \right] e^{-2\pi i \xi x} dx \\
&= \int_{\mathbb{R}^d} g(y) \int_{\mathbb{R}^d} f(x-y) e^{-2\pi i \xi z} e^{-2\pi i \xi y} dz dy \\
&= \int_{\mathbb{R}^d} g(y) e^{-2\pi i \xi y} \left(\int_{\mathbb{R}^d} f(x-y) e^{-2\pi i \xi z} dz \right) dy \\
&= \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

upon making the change of variable $z = x - y, dz = dx$. Now, for boundedness, since from Euler formula

$$|e^{i\xi 2\pi x}| = |\cos(2\pi\xi x) + i\sin(2\pi\xi x)| \leq 2$$

By the triangle inequality,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}dx\right| &\leq \int_{\mathbb{R}^d}\left|f(x)e^{-2\pi i x\xi}dx\right| \\ &\leq \int_{\mathbb{R}^d}|f(x)|\left|e^{-2\pi i x\xi}dx\right| \\ &\leq 2\int_{\mathbb{R}^d}|f(x)|dx < \infty\end{aligned}$$

since by assumption $f(x) \in L^1(\mathbb{R}^d)$.

For continuity, we use the sequential definition of continuity, letting ξ_n be defined as a sequence of points $\xi_n \to \xi$ as $n \to \infty$. Now

$$\begin{aligned}\left|\hat{f}(\xi_n) - \hat{f}(\xi)\right| &= \left|\int_{\mathbb{R}^d} f(x)(e^{-2\pi i\xi_n x} - e^{-2\pi i\xi x})dx\right| \\ &\leq \int_{\mathbb{R}^d}|f(x)|\left|(e^{-2\pi i\xi_n x} - e^{-2\pi i\xi x})\right|dx\end{aligned}$$

using the triangle inequality. Since the function inside the integral; is dominated by $2\|f\|_{L^1}$, we can use the Dominated convergence theorem; thus

$$\begin{aligned}\lim_{n\to\infty}|\hat{f}(\xi_n) - \hat{f}(\xi)| &\leq \lim_{n\to\infty}\int_{\mathbb{R}^d}|f(x)|\left|(e^{-2\pi i\xi_n x} - e^{-2\pi i\xi x})\right|dx \\ &= \int_{\mathbb{R}^d}|f(x)|\lim_{n\to\infty}\left|(e^{-2\pi i\xi_n x} - e^{-2\pi i\xi x})\right|dx = 0\end{aligned}$$

which proves continuity. This completes the proof of (e).

22. Prove that if $f \in L^1(\mathbb{R}^d)$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}dx,$$

then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. (This is the Riemann-Lebesgue lemma).
[Hint: Write $\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^d}\left[f(x) - f(x-\xi')\right]e^{-2\pi i x\xi}dx$, where $\xi' = \frac{1}{2}\frac{\xi}{|\xi|^2}$, and use Proposition 2.5.]

Proof Using the hint, we first start with $\hat{f}(\xi)$, which we translate by a factor of ξ'. Since the Lebesgue integral is invariant under translation, we get that

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^d} f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)e^{-2\pi i x\xi + \frac{2\pi i\xi^2}{2|\xi|^2}}dx \\ &= \int_{\mathbb{R}^d} f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)e^{-2\pi i x\xi}e^{\frac{2\pi i\xi^2}{2|\xi|^2}}dx \\ &= -\int_{\mathbb{R}^d} f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)e^{-2\pi i x\xi}dx\end{aligned}$$

since $e^{\pi i} = \cos(\pi i) + i\sin(\pi i) = -1$ using Euler's identity. Thus, since the above formulation, which I label $\hat{f}^*(\xi)$ is equal to the original formulation, we can use the hint and rewrite it in terms of

$$\hat{f}(\xi) = \frac{1}{2}[\hat{f}(\xi) - \hat{f}^*(\xi)]$$
$$= \frac{1}{2}\int_{\mathbb{R}^d}[f(x) - f(x-\xi')]e^{-2\pi i x\xi}dx$$

So we have using the triangle inequality that

$$|\hat{f}(\xi)| = \frac{1}{2}\left|\int_{\mathbb{R}^d}[f(x) - f(x-\xi')]e^{-2\pi i x\xi}dx\right|$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^d}\left|[f(x) - f(x-\xi')]e^{-2\pi i x\xi}\right|dx$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^d}\left|[f(x) - f(x-\xi')]\right|\left|e^{-2\pi i x\xi}\right|dx$$
$$\leq \int_{\mathbb{R}^d}\left|f(x) - f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)\right|dx$$

which equals $\|f(x) - f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)\|_{L^1}$. Now, using Proposition 2.5, since we can write the translation $\xi' = -\frac{1}{2}\frac{\xi}{|\xi|^2}$ that goes to zero as $|\xi| \to \infty$, we infer that $\|f(x) - f\left(x - \frac{1}{2}\frac{\xi}{|\xi|^2}\right)\|_{L^1} \to 0$ and as a consequence, from our definition, that $\hat{f}(\xi) \to 0$. ■