# MATH 354 - Assignment 6

Matthew Stevenson - 260405286

**Problem 2**

For any $\delta > 0$, let $f_\delta(x) = f(\delta x)$ for $x \in \mathbb{R}^d$. By Theorem 2.4, for any $\epsilon > 0$ given, there exists a continuous function $g\colon \mathbb{R}^d \to \mathbb{R}$ with compact support such that $\|f - g\|_{L_1} < \epsilon$. We can thus write:

$$f_\delta - f = (g_\delta - g) + (f_\delta - g_\delta) - (f - g)$$

However, $\|f_\delta - g_\delta\|_{L_1} = \|f - g\|_{L_1} < \epsilon$. Moreover, as $g$ is continuous with compact support, then:

$$\|g_\delta - g\|_{L_1} = \int_{\mathbb{R}^d} |g(\delta x) - g(x)|dx \to 0 \text{ as } \delta \to 1$$

Thus, for $|\delta - 1$ sufficiently small, we have that:

$$\|f_\delta - f\|_{L_1} \leq \|g_\delta - g\|_{L_1} + \|f_\delta - g_\delta\|_{L_1} + \|f - g\|_{L_1} < 3\epsilon$$

That is, for any $x \in \mathbb{R}^d$, $f(\delta x)$ converges to $f(x)$ in the $L^1$-norm as $\delta \to 1$.

□

**Problem 5**

**(a)** For $x, y \in \mathbb{R}$ and $t \in F$, assume WLOG that $\delta(x) > \delta(y)$ (otherwise, switch the roles of $x$ and $y$ in the inequality below), then consider the following:

$$|x - t| = |x - y + y - t| \leq |x - y| + |y - t|$$

If we take the infimum of this equation over all $t \in F$, then we have that:

$$\delta(x) = \inf_{t \in F} |x - t| \leq |x - y| + \inf_{t \in F} |y - t| = |x - y| + \delta(y) \implies \delta(x) - \delta(y) = |\delta(x) - \delta(y)| \leq |x - y|$$

Therefore, $\delta$ is Lipschitz on $\mathbb{R}$, and thus continuous on $\mathbb{R}$.

□

**(b)** Take $x \notin F$, then $\delta(x) > 0$ i.e. for any $\epsilon > 0$, $\delta(x) > \epsilon$. Now, for any $y \in F$, $\delta(y) = 0$, so we have that:

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2} dy = \int_F \frac{\delta(y)}{|x - y|^2} dy + \int_{F^C} \frac{\delta(y)}{|x - y|^2} dy = \int_{F^C} \frac{\delta(y)}{|x - y|^2} dy$$

If we consider the ball $B(x, \frac{\epsilon}{2})$ centered at $x$ of radius $\frac{\epsilon}{2}$, then we must have that $\delta(y) < \frac{\epsilon}{2}$. Moreover, by the continuity of $\delta$, we can make $|x - y| \leq \frac{\epsilon}{2}$. Thus, consider the following:

$$\int_{F^C} \frac{\delta(y)}{|x - y|^2} dy \geq \int_{F^C \cap B(x, \frac{\epsilon}{2})} \frac{\delta(y)}{|x - y||x - y|} dy \geq \int_{F^C \cap B(x, \frac{\epsilon}{2})} \frac{\epsilon/2}{|x - y| \cdot \epsilon/2} dy = \int_{F^C \cap B(x, \frac{\epsilon}{2})} \frac{1}{|x - y|} dy = \infty$$

To see that this last integral is infinite, we just use the translational invariance of the Lebesgue integral to integrate over the singularity at $x$. Therefore, for $x \notin F$, $I(x) = \infty$.

□

**(c)** For $x \in F$, $\delta(x) = 0$. Using the same argument as in (b) along with the Lipschitz condition, we have that:

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2} dy \leq \int_{F^C} \frac{\delta(y)}{|x - y||\delta(x) - \delta(y)|} dy \int_{F^C} \frac{\delta(y)}{|x - y||0 - \delta(y)|} dy = \int_{F^C} \frac{1}{|x - y|} dy$$

If $x \notin \partial F$, then this is the integral of a continuous function over a set of finite measure, and thus the integral $\int_{F^C} \frac{1}{|x-y|} dy < \infty$.

If, however, $x \in \partial F$, then we the above integral may not converge. If we consider the open set $F^C \subset \mathbb{R}$., there must exist a countable set $\{I_k\}_{k=1}^\infty$ of disjoint open intervals such that $F^C = \bigcup_{k=1}^\infty I_k$. The union of all boundary points of the $I_k$'s are exactly the boundary points of $F$, i.e. we have that $\partial F = \bigcup_{k=1}^\infty \partial(I_k)$. However, each $\partial(I_k)$ is at most 2 points in $\mathbb{R}$, so $\partial F$ is a countable subset of $\mathbb{R}$, i.e. $\partial F$ is a set of measure zero. In otherwords, we don't care what happens for $x \in \partial F$.

Therefore, $I(x) \leq \int_{F^C} \frac{1}{|x-y|} dy < \infty$ for $x \in F$ and not on the boundary, i.e. not in a set of measure zero. Thus, $I(x) < \infty$ for a.e. $x \in F$.

□

**Problem 6**

**(a)** Let $f$ be the the function defined as follows:

$$f(x) = \begin{cases} n & : x \in [n, n + \frac{1}{n^3}) \text{ for } n \geq 2 \\ 0 & : \text{ otherwise} \end{cases}$$

In order to continuously extend $f$, we can interpolate linearly from 0 to $n$ for $x \in [n - \frac{1}{n^3}, n]$ and similarly from $n$ to 0 for $x \in [n + \frac{1}{n^3}, n + \frac{2}{n^3}]$. Note that $\frac{1}{n^3}$ is fairly arbitrary, it just needs to be sufficiently small. In effect, we have added to $f$ the hypotenuse of the triangle formed by $n$ and 0.

Let $g$ be this continuous extension of $f$, then consider the integral of $g$ over $\mathbb{R}$:

$$\begin{aligned} \int_{\mathbb{R}} g(x)dx &= \int_{\mathbb{R}} f(x)dx + (\text{area under the triangles added}) \\ &= \sum_{n=2}^{\infty} n \cdot (n + \frac{1}{n^3} - n) + 2\sum_{n=2}^{\infty}\left(\frac{n \cdot 1/n^3}{2}\right) \\ &= \sum_{n=2}^{\infty} \frac{1}{n^2} + \sum_{n=2}^{\infty} \frac{1}{n^2} \\ &= 2\left(\frac{\pi^2}{6} - 1\right) \\ &< \infty \end{aligned}$$

Therefore, since $|g| = g$, $g$ is integrable. Furthermore, it is clear that $g$ is zero almost everywhere, so $\limsup_{x\to\infty} g(x)$. Thus, there does indeed exists a positive continuous functions that is integrable on $\mathbb{R}$ yet whose lim sup is zero.

□

**Problem 7**
Notice that $\Gamma$ is a $d$-manifold in $\mathbb{R}^{d+1}$, so any sufficiently small neighborhood on $\Gamma$ is locally homeomorphic to $\mathbb{R}^d$. If we construct a countable partition $\{Q_j\}_{j=1}^{\infty}$ of $\mathbb{R}^d$ into almost disjoint closed cubes, then this induces a partition $\{\tilde{Q}_j\}_{j=1}^{\infty}$ of $\Gamma$. Since each $\tilde{Q}_j$ is homeomorphic to $\mathbb{R}^d$ as a subspace of $\mathbb{R}^{d+1}$, the $(d+1)^{\text{st}}$-dimensional measure of $\tilde{Q}_j$ is zero, i.e. $m(\tilde{Q}_j) = 0$. By countable subadditivity, we have that:

$$m(\Gamma) \leq \sum_{j=1}^{\infty} m(\tilde{Q}_j) = \sum_{j=1}^{\infty} 0 = 0$$

Clearly, $Q = \bigcup_{j=1}^{\infty} \tilde{Q}_j$ is a measurable set of measure zero, and $\Gamma \subset Q$, so $\Gamma$ is measurable and has measure zero.

□

**Problem 9**
Since $f \geq 0$ and for $\alpha > 0$, $E_\alpha = \{x : f(x) > \alpha\} \subset \mathbb{R}^d$, then $\int_{\mathbb{R}^d} f \geq \int_{E_\alpha} f$, by monotonicity. Moreover, on $E_\alpha$, $f > \alpha$, so:

$$\int_{\mathbb{R}^d} f \geq \int_{E_\alpha} f \leq \int_{E_\alpha} \alpha = \alpha \cdot m(E_\alpha) \Longrightarrow m(E_\alpha) \leq \frac{1}{\alpha}\int_{\mathbb{R}^d} f$$

□

**Problem 14**
(a) Consider the function $f(x) = 2(1-x^2)^{1/2}$. It is clearly $\mathbb{R}$-measurable, so by Corollary 3.8, $B_1 = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}$ is $\mathbb{R}^2$-measurable, and moreover, we have that:

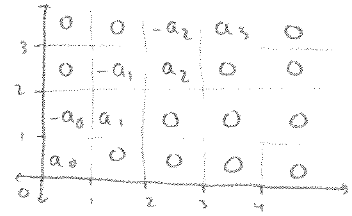
$$v_2 = m(B_1) = \int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} 2(1-x^2)^{1/2} \cdot \chi_{[-1,1]}dx = 2\int_{-1}^{1}(1-x^2)^{1/2}dx$$

We can evaluate this integral by using the substitution $x = \sin(t)$ as follows:

$$v_2 = 2\int_{-\pi/2}^{\pi/2}(1-\sin^2(t))^{1/2}\cos(t)dt = 2\int_{-\pi/2}^{\pi/2}\cos^2(t)dt = t + \sin(t)\cos(t)\Big|_{-\pi/2}^{\pi/2} = \pi$$

□

**Problem 17**
(a) We can draw $\mathbb{R}^2$ where the value of $f$ is indicated in each region:



If we first fix $x$, then consider the following:

$$\int_{\mathbb{R}} |f|^x(y)dy = \begin{cases} a_{\lfloor x \rfloor} + a_{\lfloor x \rfloor} + 0 + \cdots & : x \geq 0 \\ 0 & : \text{otherwise} \end{cases} < \infty$$

Therefore, for fixed $x$, $f^x$ is integrable. Similarly, if we fix $y$, consider the following:

$$\int_{\mathbb{R}} |f|^y(x)dx = \begin{cases} a_{\lfloor y \rfloor -1} + a_{\lfloor y \rfloor -2} + 0 + \cdots & : y \geq 1 \\ a_0 & : y \in [0,1) \\ 0 & : \text{otherwise} \end{cases} < \infty$$

Therefore, for fixed $y$, $f^y$ is integrable. If we now compute the integral of $f^x$ for a fixed $x > 0$, we have that:

$$\int_{\mathbb{R}} f^x(y)dy = a_{\lfloor x \rfloor} - a_{\lfloor x \rfloor} + 0 + \cdots = 0$$

We can thus conclude that:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dydx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f^x(y)dy \right) dx = \int_{\mathbb{R}} 0 \cdot dx = 0$$

□

**(b)** From the diagram above, we can see that for a fixed $y \in (0, \infty)$:

$$\int_{\mathbb{R}} f^y(x)dx = \begin{cases} a_0 & : y \in [0,1) \\ a_n - a_{n-1} & : y \in [n, n+1) \text{ for } n \geq 1 \\ 0 & : \text{otherwise} \end{cases}$$

It is clear that the integral of $|f^y|$ for fixed $y \in (0,\infty)$ is finite, so $y \mapsto \int f^y(x)dx$ is integrable on $(0,\infty)$. Now, consider the following:

$$\begin{aligned} \int_{(0,\infty)} \left( \int_{\mathbb{R}} f^y(x)dx \right) dy &= \int_0^1 a_0 dy + \int_1^2 (a_1 - a_0)dy + \int_2^3 (a_2 - a_1)dy + \cdots \\ &= \int_0^1 a_0 dy + \sum_{k=1}^{\infty} \int_k^{k+1} (a_k - a_{k-1})dy \\ &= a_0 \cdot (1-0) + \sum_{k=1}^{\infty} (a_k - a_{k-1}) \cdot (k+1-k) \\ &= b_0 + \sum_{k=1}^{\infty} b_k \\ &= s \end{aligned}$$

□

**(c)** We note that the integral over $\mathbb{R}^2$ of $|f|$ is equivalent to summing the absolute value of each square in the diagram above:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x,y)|dxdy = \sum_{i=0}^{\infty} 2a_i = 2\sum_{i=0}^{\infty} \left( \sum_{k=0}^{i} b_i \right) = \infty$$

We clearly have that this sum diverges as we are counting every value $b_k$ infinitely-many times, and taking the sum of these infinite sums of positive numbers. Thus, $|f|$ is not integrable (which is why Fubini's Theorem cannot be applied).

□

**Problem 19**

Assume WLOG that $f$ is positive on $\mathbb{R}^d$ and let $\alpha > 0$. The set $E_\alpha = \{x \in \mathbb{R}^d : f(x) > \alpha\}$ is the complement of the preimage $f^{-1}[-\infty, \alpha)$, i.e. the complement of a measurable set. Thus, $E_\alpha$ is measurable as $f$ is integrable, and so we have that:

$$\int_0^{\infty} \chi_{E_\alpha}(x)dx = m(E_\alpha)$$

Moreover, for $x \in \mathbb{R}^d$, we can rewrite $f(x)$ as follows:

$$f(x) = \int_0^{\infty} \chi_{[0,f(x))}(t)dt$$

Thus, using Fubini's Theorem, we have that:

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d}\int_0^{\infty} \chi_{[0,f(x))}(t)dtdx = \int_0^{\infty}\int_{\mathbb{R}^d} \chi_{[0,f(x))}(t)dxdt = \int_0^{\infty} m(\{x \in \mathbb{R}^d : f(x) > t\})dt = \int_0^{\infty} m(E_t)dt$$

□

**Problem 21**

(a) If $f, g$ are measurable on $\mathbb{R}^d$, then for any $x, y \in \mathbb{R}^d$, the preimages $f^{-1}[-\infty, x)$ and $g^{-1}[-\infty, y)$ are measurable sets. In particular, for $x, x-y \in \mathbb{R}^d$, we have that the preimage of $f(x-y)g(y)$ is $f^{-1}[-\infty, x-y) \times g^{-1}[-\infty, y)$, and the Cartesian product of measurable sets is measurable in $\mathbb{R}^{2d}$. Therefore, any preimage of $f(x-y)g(y)$ is a measurable set in $\mathbb{R}^{2d}$ i.e. $f(x-y)g(y)$ is a measurable function on $\mathbb{R}^{2d}$.

□

(b) For $f, g \in L_1(\mathbb{R}^d)$, by translation invariance of the Lebesgue integral, then:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|dxdy = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x)|dx\right)|g(y)|dy = \int_{\mathbb{R}^d}\|f\|_{L_1}|g(y)|dy = \|f\|_{L_1}\int_{\mathbb{R}^d}|g(y)|dy = \|f\|_{L_1}\|g\|_{L_1} < \infty$$

Therefore, $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

□

(c) Since $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$, we have that:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x-y)g(y)|dydx < \infty \implies \int_{\mathbb{R}^d}|f(x-y)g(y)|dy < \infty \text{ for a.e. } x \in \mathbb{R}^d$$

Thus, $f(x-y)g(y)$ is integrable for a.e. $x \in \mathbb{R}^d$ i.e. $(f*g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^d$.

□

(d) For $f, g \in L_1(\mathbb{R}^d)$, then using Fubini's Theorem and the argument from (b), we have that:

$$\|f*g\|_{L_1} = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(x-y)g(y)dx\right|dy \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x-y)g(y)|dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x-y)g(y)|dydx = \|f\|_{L_1}\|g\|_{L_1} < \infty$$

Therefore, $f*g \in L_1(\mathbb{R}^d)$, as required. Notice that if $f, g$ are nonnegative on $\mathbb{R}^d$, then the only inequality above must be an equality; that is, $\|f*g\|_{L_1} = \|f\|_{L_1}\|g\|_{L_1}$.

□

(e) For $f \in L_1(\mathbb{R}^d)$, $\|f\|_{L_1} < \infty$; so notice that:

$$|\hat{f}(\xi)| = \left|\int_{\mathbb{R}^d} f(x)e^{-i\xi x}dx\right| \le \int_{\mathbb{R}^d}|f(x)||e^{-i\xi x}|dx = \int_{\mathbb{R}^d}|f(x)|dx = \|f\|_{L_1} < \infty$$

Therefore, $\hat{f}(\xi)$ is bounded. Now, let $\xi_n$ be a sequence in $\mathbb{R}$ such that $\xi_n \to \xi$ as $n \to \infty$. Consider the following:

$$|\hat{f}(\xi_n) - \hat{f}(\xi)| = \left|\int_{\mathbb{R}^d} f(x)(e^{-i\xi_n x} - e^{-i\xi x})dx\right| \le \int_{\mathbb{R}^d}|f(x)||e^{-i\xi_n x} - e^{-i\xi x}|dx$$

The integrand of the above function is clearly dominated by $2|f(x)|$ for any $x \in \mathbb{R}^d$, and we clearly have that $2|f| \in L_1(\mathbb{R}^d)$ since $f \in L_1(\mathbb{R}^d)$. Thus, by the Dominated Convergence Theorem, we have that:

$$\lim_{n\to\infty}|\hat{f}(\xi_n) - \hat{f}(\xi)| \le \lim_{n\to\infty}\int_{\mathbb{R}^d}|f(x)||e^{-i\xi_n x} - e^{-i\xi x}|dx = \int_{\mathbb{R}^d}|f(x)| \cdot \lim_{n\to\infty}|e^{-i\xi_n x} - e^{-i\xi x}|dx = \int_{\mathbb{R}^d}|f(x)| \cdot 0 \cdot dx = 0$$

Therefore, using its sequential definition, $\hat{f}$ is continuous on $\mathbb{R}$.

For $f, g \in L_1(\mathbb{R}^d)$, then using Fubini's Theorem, we have that:

$$(\widehat{f*g})(\xi) = \int_{\mathbb{R}^d}(f*g)(x)e^{-i\xi x}dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x-y)g(y)dy\right)e^{-i\xi x}dx = \int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} f(x-y)e^{-i\xi x}dxdy$$

If we use the substitution $u = x - y$ in the inner integral, then we have:

$$\int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} f(x-y)e^{-i\xi x}dxdy = \int_{\mathbb{R}^d} g(y)\int_{\mathbb{R}^d} f(u)e^{-i\xi u}e^{-i\xi y}dudy = \left(\int_{\mathbb{R}^d} g(y)e^{-i\xi y}dy\right)\left(\int_{\mathbb{R}^d} f(u)e^{-i\xi u}du\right) = \hat{f}(\xi)\hat{g}(\xi)$$

Therefore, we have that $(\widehat{f*g})(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

□

**Problem 22**

Let $\xi' = \frac{1}{2}\frac{\xi}{|\xi|^2}$, then $\xi' \to 0 \iff |\xi| \to 0$. Moreover, we can write:

$$|\hat{f}(\xi)| = \left|\frac{1}{2}\int_{\mathbb{R}^d}(f(x) - f(x-\xi'))e^{-2\pi i\xi x}dx\right| \le \frac{1}{2}\int_{\mathbb{R}^d}|f(x) - f(x-\xi')| \cdot |e^{-2\pi i \xi x}|dx$$

By Proposition 2.5, we know that $\|f_{\xi'} - f\|_{L_1} \to 0$ as $\xi' \to 0$. Thus, for any $x \in \mathbb{R}^d$, $|f(x) - f(x-\xi')| \to 0$ as $\xi' \to 0 \iff |\xi| \to 0$. We thus have that:

$$\frac{1}{2}\int_{\mathbb{R}^d}|f(x) - f(x-\xi')| \cdot |e^{-2\pi i\xi x}|dx \to \frac{1}{2}\int_{\mathbb{R}^d} 0 \cdot dx = 0 \text{ as } |\xi| \to 0$$

Therefore, $\hat{f}(\xi) \to 0$ as $|\xi| \to 0$.

□