Yvan Le
ID#: 260 430 278
MATH-354
Assignment #5

Exercise 2.

b) To show that F is well-defined, it is enough to show that $\forall x \in C$, its corresponding ternary expansion $x = \sum_{k\in\mathbb{N}} \frac{a_k}{3^k}$ in which $a_k \in \{0,2\}$ is unique. Suppose not; let $x \in C$ be such that it has two such distinct ternary expansions, denoted $\sum_{k\in\mathbb{N}} \frac{a_k}{3^k}$ and $\sum_{k\in\mathbb{N}} \frac{a'_k}{3^k}$. Since these ternary expansions are distinct, then $\exists k \in \mathbb{N}$ such that $a_k \neq a'_k$; let k_0 be the smallest such k. Then $0 = x - x = \sum_{k\in\mathbb{N}} \frac{c_k}{3^k}$ where for any $k \in \mathbb{N}, c_k = a_k - a'_k$. Since for $k < k_0$ we have $a_k = a'_k$ then $c_k = 0$, and so we have $0 = \sum_{k\geq k_0} \frac{c_k}{3^k}$. Now we note that $c_{k_0} \in \{-2,2\}$. Furthermore, $\forall k > k_0, c_k \in \{-2,0,2\}$.

Suppose that $c_{k_0} = -2$.
Then $0 = -\frac{2}{3^{k_0}} + \sum_{k>k_0} \frac{c_k}{3^k} \leq -\frac{2}{3^{k_0}} + \sum_{k>k_0} \frac{2}{3^k} = -\frac{2}{3^{k_0}} + 2\sum_{k>k_0} \frac{1}{3^k} = -\frac{2}{3^{k_0}} + \frac{1}{3^{k_0}} = -\frac{1}{3^{k_0}}$.
That is, $0 \leq -\frac{1}{3^{k_0}}$. Clearly, this is a contradiction.

Now, suppose that $c_{k_0} = 2$.
Then $0 = \frac{2}{3^{k_0}} + \sum_{k>k_0} \frac{c_k}{3^k} \geq \frac{2}{3^{k_0}} + \sum_{k>k_0} \frac{(-2)}{3^k} = \frac{2}{3^{k_0}} - 2\sum_{k>k_0} \frac{1}{3^k} = \frac{2}{3^{k_0}} - \frac{1}{3^{k_0}} = \frac{1}{3^{k_0}}$.
That is, $0 \geq \frac{1}{3^{k_0}}$; again, this is a contradiction. Hence, we conclude that for any $x \in C$, its ternary expansion with $a_k \in \{0,2\}$ is unique, and so F is well-defined.

Now, to show continuity, we first note that F is monotone non-decreasing; let $x, y \in C$ be such that $x < y$ with ternary expansions $x = \sum_{k\in\mathbb{N}} \frac{a_k}{3^k}, y = \sum_{k\in\mathbb{N}} a'_k/3^k$. Since $x < y$, then for some $k_0 \in \mathbb{N}$, if $k < k_0$ then $a_k = a'_k$, and $a_{k_0} = 0, a'_{k_0} = 2$.

Denote $F(x) = \sum_{k\in\mathbb{N}} \frac{b_k}{2^k}$ and $F(y) = \sum_{k\in\mathbb{N}} \frac{b'_k}{2^k}$; then for $k < k_0$, we have $b_k = b'_k$ and $b_{k_0} = 0, b'_{k_0} = 1$, and so $F(x) \leq F(y)$ (with equality if and only if $b_k = 1$ and $b'_k = 0$ for any $k > k_0$). So F is monotone non-decreasing.

Let $\epsilon > 0$ be given; w.l.o.g., we may assume that $\epsilon < 1$. We note that $\exists k \in \mathbb{N}$ such that $\frac{1}{2^k} \leq \epsilon$; let k_0 be the smallest such k. Pick $\delta = \frac{1}{3^{k_0}}$ and fix $x \in C$, where $x = \sum_{k\in\mathbb{N}} \frac{a_k}{3^k}$ and $\forall k \in \mathbb{N}, a_k \in \{0,2\}$. We note that $\forall y \in C$ such that $|x - y| < \delta$, we have two bounds on y given by $y \leq y_1 := \sum_{k=1}^{k_0} \frac{a_k}{3^k} + \sum_{k>k_0} \frac{2}{3^k}$ and $y \geq y_2 := \sum_{k=1}^{k_0} \frac{a_k}{3^k} + \sum_{k>k_0} \frac{0}{3^k}$. By monotonicity, it is sufficient to show that $F(y_1) \leq F(x) + \frac{1}{2^{k_0}}$ and $F(y_2) \geq F(x) - \frac{1}{2^{k_0}}$. Denote $F(x) = \sum_{k\in\mathbb{N}} \frac{b_k}{2^k}$ where $b_k = \frac{a_k}{2}$. Then $F(y_1) = \sum_{k=1}^{k_0} \frac{b_k}{2^k} + \sum_{k>k_0} \frac{1}{2^k}$; that is, $F(y_1) = \sum_{k=1}^{k_0} \frac{b_k}{2^k} + \frac{1}{2^{k_0}} \leq \sum_{k\in\mathbb{N}} \frac{b_k}{2^k} + \frac{1}{2^{k_0}} = F(x) + \frac{1}{2^{k_0}}$ as claimed.

Similarly, $F(x) - \frac{1}{2^{k_0}} = \sum_{k\in\mathbb{N}} \frac{b_k}{2^k} - \sum_{k>k_0} \frac{1}{2^k} \le \sum_{k=1}^{k_0} \frac{b_k}{2^k} + \sum_{k>k_0} \frac{0}{2^k} = F(y_2)$ as claimed. Hence, F is continuous on C. Finally, $0 = \sum_{k\in\mathbb{N}} \frac{0}{3^k} \to F(0) = \sum_{k\in\mathbb{N}} \frac{0}{2^k} = 0$ and $1 = \sum_{k\in\mathbb{N}} \frac{2}{3^k} \to F(1) = \sum_{k\in\mathbb{N}} \frac{1}{2^k} = 1$ as claimed. ■

c) We note that $\forall y \in [0,1]$, y has a binary expansion; that is, $y = \sum_{k\in\mathbb{N}} \frac{b_k}{2^k}$ where $b_k \in \{0,1\}$. Pick any such expansion, and define $x = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$ where $a_k = 2b_k$; then clearly, $F(x) = y$ and since $2b_k \in \{0,2\}$, then x has a ternary expansion where $a_k \in \{0,2\}$ and so $x \in C$. Hence, F is surjective. ■

d) Suppose $(a,b) \subseteq [0,1]\backslash C$; then $\exists n, k_0 \in \mathbb{N}$ such that $a = \frac{n}{3^{k_0}}$ and $b = \frac{n+1}{3^{k_0}}$, where both n and $n+1$ are co-prime to 3. So $n \equiv 1(mod\ 3)$ and $n+1 \equiv 2(mod\ 3)$. Since $n \equiv 1(mod\ 3)$, let $a = \sum_{k=1}^{k_0} \frac{a_k}{3^k}$, where $a_{k_0} = 1$ and $\forall k < k_0, a_k \in \{0,2\}$. Then $a = \sum_{k=1}^{k_0-1} \frac{a_k}{3^k} + \frac{1}{3^{k_0}} = \sum_{k=1}^{k_0-1} \frac{a_k}{3^k} + \frac{0}{3^{k_0}} + \sum_{k>k_0} \frac{2}{3^k}$. On the other hand, we have $b = \sum_{k=1}^{k_0-1} \frac{a_k}{3^k} + \frac{2}{3^{k_0}}$. Let $F(a) = \sum_{k\in\mathbb{N}} \frac{b_k}{2^k}$ with $b_k = \frac{a_k}{2}$. $a_{k_0} = 0$, so $b_{k_0} = 0$; furthermore, $\forall k > k_0, a_k = 2$ so $b_k = 1$. So $F(a) = \sum_{k=1}^{k_0-1} \frac{b_k}{2^k} + \frac{0}{2^{k_0}} + \sum_{k>k_0} \frac{1}{2^k} = \sum_{k=1}^{k_0-1} \frac{b_k}{2^k} + \frac{1}{2^{k_0}}$. On the other hand, $F(b) = \sum_{k=1}^{k_0-1} \frac{b_k}{2^k} + \frac{1}{2^{k_0}} = F(a)$ as claimed. Hence, the function F obtained by extending to $[0,1]$ as described is well-defined.

Continuity is also clear; for any $x \in Int(C)$ (the interior of C), F is continuous at x as shown in b). Furthermore, for any $x \in [0,1]\backslash C$, clearly F is continuous at x, since the function is constant on a sufficiently small neighborhood centered at x. Hence, it remains to be shown that F is continuous at any $x \in \partial C$, the boundary of C. Suppose w.l.o.g. that x is the left end-point of some sufficiently small closed interval contained in C (here we allow singleton points to be a closed interval), and that x is the right end-point of some sufficiently small open interval contained $[0,1]\backslash C$. By continuity of F in C, we have $\lim_{t\to x^+} F(t) = F(x)$. By continuity of F in $[0,1]\backslash C$, we have $\lim_{t\to x^-} F(t) = F(x)$, and so F is continuous everywhere on $[0,1]$. ■

Exercise 14.

a) We note that by analogy with $m_*(E)$, the intervals I_j covering E as defined for $J_*(E)$ are closed. Let $\{I_j\}_{j=1}^{N}$ be a finite covering of E by closed intervals; we claim that any limit point of E is covered as well. Let x be a limit point of E, and let $(x_n)_{n=1}^{\infty}$ be a sequence in E converging to x. Since $E \subseteq \cup_{j=1}^{N} I_j$, then $(x_n)_{n=1}^{\infty}$ is a sequence in $\cup_{j=1}^{N} I_j$. Since $\cup_{j=1}^{N} I_j$ is a finite union of closed intervals, then it must be closed as well, and so it contains all its limit points; hence, $x \in \cup_{j=1}^{N} I_j$. Therefore, it follows that $\bar{E} \subseteq \cup_{j=1}^{N} I_j$. Then clearly, for any $\{I_j\}_{j=1}^{N}$ covering $\bar{E}$, $\{I_j\}_{j=1}^{N}$ must cover E. Hence, $J_*(E) = J_*(\bar{E})$. ■

b) Consider $E := \mathbb{Q} \cap [0,1]$, the set of rationals in $[0,1]$. Since this set is countable, $m_*(E) = 0$; however, since $\bar{E} = [0,1]$, then $J_*(E) = 1$. ■

Exercise 16.

a) We have $E = \cap_{n=1}^{\infty} \cup_{k\geq n} E_k$. For any $n \in \mathbb{N}$, let $E_n' = \cup_{k\geq n} E_k$; since it is a countable union of measurable sets, then E_n' is measurable. Then $E = \cap_{n=1}^{\infty} E_n'$ is a countable intersection of measurable sets, and so E is measurable. ■

b) Since we have $\sum_{k=1}^{\infty} m(E_k) < \infty$, then for any $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\sum_{k\geq N} m(E_k) < \epsilon$. By countable sub-additivity, this implies then that $m(\cup_{k\geq N} E_k) \leq \sum_{k\geq N} m(E_k) < \epsilon$. We note that $E \subseteq \cup_{k\geq N} E_k$ and so by monotonicity, $m(E) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $m(E) = 0$. ■

Exercise 21.

Consider $F: \mathcal{C} \to [0,1]$ as defined in Problem 2, and consider the non-measurable set $\mathcal{N} \subseteq [0,1]$ as defined on p.24 (Stein). Since F is surjective, then $F^{-1}(\mathcal{N})$ is well-defined, and we have $F^{-1}(\mathcal{N}) \subseteq \mathcal{C}$. By monotonicity of the exterior measure, $m_*(F^{-1}(\mathcal{N})) \leq m_*(\mathcal{C})$. Since the Cantor set is Lebesgue measurable with measure 0, we have $m_*(\mathcal{C}) = m(\mathcal{C}) = 0$, and so $m_*(F^{-1}(\mathcal{N})) = 0$. Sets of exterior measure 0 are measurable, and so we conclude that $F^{-1}(\mathcal{N})$ is measurable. Hence, this is an example of a continuous function that maps a measurable set to a non-measurable set. ■

Exercise 28.

Fix $0 < \alpha < 1$; let $O \subseteq \mathbb{R}$ be an open set such that $E \subseteq O$ and $m_*(E) \geq \alpha m_*(O)$. Let $O = \coprod_{k=1}^{\infty} I_k$ where I_k are open intervals in $\mathbb{R}$. We note that $E \cap O = E \cap \coprod_{k=1}^{\infty} I_k = \coprod_{k=1}^{\infty}(E \cap I_k)$. Furthermore, since $E \subseteq O$, we have $E \cap O = E$; hence, $E = \coprod_{k=1}^{\infty}(E \cap I_k)$. Now, suppose for contradiction that $\forall k \in \mathbb{N}$ we have $m_*(E \cap I_k) < \alpha m_*(I_k)$. Since I_k's are disjoint and measurable, we must have $m_*(E) \leq m_*(\coprod_{k=1}^{\infty}(E \cap I_k)) = \sum_{k=1}^{\infty} m_*(E \cap I_k) < \sum_{k=1}^{\infty} \alpha m_*(I_k) = \alpha \sum_{k=1}^{\infty} m_*(I_k) = \alpha m_*(O)$; where the first inequality follows from countable sub-additivity. That is, $m_*(E) < \alpha m_*(O)$. This is a contradiction, and so $\exists I_k$ such that $m_*(E \cap I_k) \geq \alpha m_*(I_k)$. ■

Exercise 29.

By Exercise 28, $\exists I \subseteq R$ open such that $m(E \cap I) \geq \frac{9}{10} m(I)$. Let $E_0 = E \cap I$. Let $I = (a,b)$; then $m(I) = b - a$. We claim that if the difference set of E_0 contains an interval centered at the origin, then so must E. Suppose such an interval exists for E_0, denote $(-a,a)$. Then $\forall x \in (-a,a), \exists y_1, y_2 \in E_0$ such that $y_1 - y_2 = x$. Since $E_0 \subseteq E$, then $\exists y_1, y_2 \in E$ such that $y_1 - y_2 = x$. Since x is arbitrary, this implies that $(-a,a)$ is contained in the difference set of E.

Suppose for contradiction that no such interval exists in the difference set of E_0. Then given $\epsilon > 0$ arbitrarily small (without loss of generality, let $\epsilon < \frac{b-a}{10}$), $\forall x, y \in E_0$ with $x > y$, we have $x - y > \epsilon \to x > y + \epsilon$. Now suppose the sets E_0 and $E_0 + \epsilon$ are not disjoint; then $\exists x, y \in E_0$ such that $x = y + \epsilon$, contradicting the above inequality. So E_0 and $E_0 + \epsilon$ are disjoint. We note that $m(E_0 + \epsilon) = m(E_0)$, since the Lebesgue measure is invariant under translation, and so $m(E_0 \cup E_0 + \epsilon) = 2m(E_0)$.

On the other hand, $I + \epsilon = (a + \epsilon, b + \epsilon)$. Since $\epsilon < \frac{b-a}{10}$, then $I \cup I + \epsilon = (a, b + \epsilon)$, and so $m(I \cup I + \epsilon) = b + \epsilon - a < \frac{11}{10}(b - a) = \frac{11}{10}m(I)$. Since $E_0 \subseteq I$, we have $E_0 + \epsilon \subseteq I + \epsilon$ and so $E_0 \cup E_0 + \epsilon \subseteq I \cup I + \epsilon$. So $m(E_0 \cup E_0 + \epsilon) \leq m(I \cup I + \epsilon) \rightarrow 2 * \frac{9}{10}m(I) \leq 2m(E_0) \leq \frac{11}{10}m(I)$. Then $\frac{18}{10} \leq \frac{11}{10}$; this is clearly a contradiction, and so the difference set of E_0 must contain an interval centered at the origin, and so the difference set of E contains an interval centered at the origin. ■

Problem 1.

Let $E := \left\{x \in \mathbb{R}: \exists \text{ infinitely many } p/q \text{ with } \gcd(p, q) = 1 \text{ such that } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^3}\right\}$. Here, we insist that $q > 1$ and let $p \in \mathbb{Z}\backslash\{0\}$. The reasoning for letting $q > 1$ is as follows: for any $x \in \mathbb{R}$, there are finitely many $p \in \mathbb{Z}\backslash\{0\}$ such that $|x - p| \leq 1$; these p are, in fact, $\lfloor x \rfloor$ and $\lceil x \rceil$, and so the case where $q = 1$ does not affect whether or not a given x is in E. The reason for excluding $p = 0$ is simply because 0 is not relatively prime to any number other than 1.

First, we let $\mathbb{R} = \coprod_{n \in \mathbb{Z}}[n, n + 1)$ and denote $E_n := E \cap [n, n + 1)$; then clearly, $E = \coprod_{n \in \mathbb{Z}} E_n$. To show that $m(E) = 0$, it is sufficient to show that $\forall n \in \mathbb{Z}$, E_n is measurable and $m(E_n) = 0$. Fix $n \in \mathbb{Z}$.
For any $q > 1$, define $E_{n,q} := \left\{x \in [-n, n): \exists p \in \mathbb{Z}\backslash\{0\} \text{ such that } \gcd(p, q) = 1 \text{ and } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^3}\right\}$. Then $E_n := \{x \in [-n, n): x \in E_{n,q} \text{ for infinitely many } q\}$. That is, $E_q := \limsup_{k \to \infty}(E_{q,k})$.

Fix $q > 1$. $x \in E_{n,q} \rightarrow x \in [-n, n)$ and $\exists p \in \mathbb{Z}\backslash\{0\}$ such that $\gcd(p, q) = 1$ and $\left|x - \frac{p}{q}\right| \leq \frac{1}{q^3} \rightarrow$ for some $p \in \{nq + 1, nq + 2, \dots, (n + 1)q - 1\}$, $x \in \left[\frac{p}{q} - \frac{1}{q^3}, \frac{p}{q} + \frac{1}{q^3}\right]$. Note that $p \notin \{nq, (n + 1)q\}$ since we insist that p is relatively prime to q. Note that there may be some p in the given range where $\gcd(p, q) > 1$; hence, we can conclude that $E_{n,q} \subseteq \bigcup_{p=nq+1}^{(n+1)q-1} \left[\frac{p}{q} - \frac{1}{q^3}, \frac{p}{q} + \frac{1}{q^3}\right]$. Each of these intervals have length $\frac{2}{q^3}$ and there are at most $(q - 2)$ such intervals. So $m(E_{n,q}) \leq \frac{2(q-2)}{q^3} < \frac{2}{q^2}$.

Now we note that $\sum_{q \to \infty} m(E_{n,q}) < \sum_{q \to \infty} \frac{2}{q^2} < \infty$. So by the Borel-Cantelli Lemma, we conclude that $m(E_n) = 0$. Since $n \in \mathbb{Z}$ is arbitrary, we thus conclude that $m(E) = 0$. ■