Math 354
Assignment 5
Professor Jakobson

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2. b) We are given $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, $f(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$ and $b_k = \frac{a_k}{2}$ so given $|x-y| < \delta$,$x, y \in \mathcal{C}$ we find $|f(x) - f(y)| = |\sum_{k=1}^{\infty} \frac{b_{x_k}}{2^k} - \sum_{k=1}^{\infty} \frac{b_{y_k}}{2^k}| = |2\sum_{k=1}^{\infty} \frac{a_{x_k} - a_{y_k}}{2^k}| < 2|\sum_{k=1}^{\infty} \frac{a_{x_k}+a_{y_k}}{3^k}| = 2\delta$. Thus with the choice of $\delta := \frac{\epsilon}{2}$ the continuity follows. It is clear that for the ternary expansion of 0 and 1 respectively, a_k is $\{0, 0, 0...0...\}$ and $\{2, 2...2...\}$ respectively, so $f(0) = 0$ and $f(1) = \sum_{k=1}^{\infty} 2^{-k} = 1$.

c) Suppose we are given an arbitrary $y \in [0,1]$. It follows y has a binary expansion, i.e. $\exists \{c_n\}$ such that $y = \sum_{k=1}^{\infty} \frac{c_k}{2^k}$, where $c_k \in \{0,1\}$. It follows then that $a_k = 2c_k \in \{0,2\}$. Now consider our function and its mapping; $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, $f(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$. Now $a_k :=\in \{0,2\}$ and $b_k := \frac{a_k}{2}$. Notice too that with $\{b_k = c_k : \forall k\}$, $y = f(x)$ and x is an element of the cantor set. But this was an arbitrary point from $[0,1]$, so it follows that $f(x)$ is surjective.

d) Suppose we are given an open set in the complement of the cantor set, say $S = (a,b)$, and we define $\{F(x) = F(a) : x \in S\}$. Since our function is continuous in the cantor set and in the complement of the cantor set, as well as continuous at a, the continuity remains to be shown over the point b. Indeed, suppose we have a point in the delta neighbourhood of x, called y, such that $y > b$. Then $|F(y) - F(x)| \leq |F(y) - F(b)| + |F(b) - f(x)| < 2\epsilon$ so the continuity follows.

14. a) Consider a finite set Q of closed cubes which cover $cl(E)$, and a finite set of closed cubes R which cover E. It follows that since Q is a cover of $cl(E)$ it is also a cover of E, for all Q. Thus $J_*(E) = inf \sum |R| \leq inf \sum |Q| = J_*(cl(E))$. Now for the reverse inequality, consider a closed set $E_0 \subset E$. We keep a similar Q, such that Q is a set of closed cubes which cover $cl(E)$. We choose S such that $S_j + \frac{\epsilon}{2N} = Q_j$, and $\sum |S| \leq J_*(E_0) + \frac{\epsilon}{2}$. It is worth noting that our first construction is viable for an adequate choice of Q. Notice that by our construction, $(\sum |S|) + \frac{\epsilon}{2} = \sum |Q|$. Then $J_*(cl(E)) \leq \sum |Q| = \sum |S| + \frac{\epsilon}{2} \leq J_*(E_0) + \epsilon \leq J_*(E) + \epsilon$ which completes the reverse inequality. We conclude $J_*(E) = J_*(cl(E))$.

b) Consider the rationals over $[0,1]$. It is true that all irrationals are limit points of rationals. Thus the closure of the rationals over $[0,1]$ is $[0,1]$. Next, $[0,1]$ covers itself, so $J_*([0,1]) \leq 1$. Also, $J_*([0,r] \leq r$ by the same logic. Since $[0,r] \subset [0,1]|r < 1$, it follows $r < J_*([0,1]) \leq 1$. But r is arbitrarily smaller than 1, so it follows $1 \leq J_*([0,1]) \leq 1$, so $J_*([0,1]) = 1$. But by a), it follows the jordan content of the rationals over $[0,1]$ is 1. Also, because the rationals are countable, we may create a countable number of disjoint sets each with length $\frac{\epsilon}{2^k}$. It follows from the sub-additivity property of the lebesgue measure that $m(E) \leq \sum_{k=1}^{\infty} m(E_k) = \epsilon$. Since ϵ is arbitrary, we conclude $m(E) = 0$.

16. a) We know that $E = \cap_{k=1}^{\infty} \cup_{n\geq k}^{\infty} E_n$, and all E_k are measurable. Since measure is closed under countable union, $F_k = \cup_{n\geq k}^{\infty} E_n$ is measurable for any k. Now measure is closed under countable intersection, so $\cap_{k=1}^{\infty} F_k$ is measurable. It follows E is a measurable set.

by induction!

b) $\cap_{k=1}^{\infty} \cup_{n\geq k}^{\infty} E_n$. Consider the case $k = 1, k = 2$, then $(E_1 \cup E_2 \cup ... \cup E_i ...) \cap (E_2 \cup E_3 \cup ... \cup E_i ...) = \cup_{n\geq 2}^{\infty} E_n$. Now consider $k = k'$ such that $\cap_{k=k}^{k'\infty} \cup_{n\geq k}^{\infty} E_n = \cup_{k'}^{\infty} E_n$, then $\cup_{k'}^{\infty} E_n \cap \cup_{k'+1}^{\infty} E_n = (E_{k'} \cup E_{k'+1} \cup ... \cup E_i \cup ...) \cap (E_{k+1} \cup E_{k+1} \cup ... \cup E_i \cup ...) = \cup_{n=k'+1}^{\infty} E_n$. Thus $E = \cap_{k=1}^{\infty} \cup_{n\geq k}^{\infty} E_n = \cup_{n=k}^{\infty} E_n$. By the sub-additivity of the measure, $0 \leq m(E) \leq \sum_{n=k}^{\infty} m(E_n)$. But it is clear, since their sum is finite, $\exists k'$ such that $\sum_{n=k'}^{\infty} m(E_n) < \epsilon$. With the choice $k = k'$ we obtain $0 \leq m(E) < \epsilon$. Since ϵ is arbitrary, it follows $m(E) = 0$.

21. Consider the function~~ed~~ defined such that $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, $a_k \in \{0,2\}$, $f(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$, and extend this function as is done in 2.d). such that $f : \mathcal{C} \to \mathcal{N}$, where $\mathcal{C}$ is the cantor set and $\mathcal{N}$ is the unmeasurable set constructed in the book and in class. We have shown the continuity of this function over $[0,1]$. Since $f : \mathcal{C} \to [0,1]$ is surjective, it follows that $f : \mathcal{C} \to \mathcal{N} \in [0,1]$ is also surjective, that is our function can truly map the cantor set to our non-measurable set. Now all that remains to be shown is the continuity of f over $f : \mathcal{C} \to \mathcal{N}$. Indeed, consider x, y such that $f(x) \in \mathcal{N}, f(y) \in \mathcal{N}, |x - y| < \delta$. Notice that $f(x), f(y) \in [0,1]$, so by the continuity of $f(x), f(y)$ over $[0,1]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < 2\epsilon$. But this immediately implies the continuity of $f : \mathcal{C} \to \mathcal{N}$, and so our proof is complete.

28. Let us choose an open set $\mathcal{O}$ such that $E \subset \mathcal{O}$ and $m_*(E) \geq \alpha m_*(\mathcal{O})$ where $\alpha \in (0,1)$. Then by the lemma 1.3, $\exists \{I_x\}$ where $I_i \cap I_j = \emptyset$, such that $\mathcal{O} = \cup_{x=1}^{\infty} I_x$. Thus $E = E \cap \cup_{x=1}^{\infty} I_x = \cup_{x=1}^{\infty} E \cap I_x$. We suppose $\nexists I_k$ such that $m_*(E \cap I_k) \geq \alpha m_*(I_k)$. It follows $\frac{m_*(E)}{\alpha} \leq \frac{\sum_{x=1}^{\infty} m_*(E \cap I_x)}{\alpha} < \sum_{x=1}^{\infty} m_*(I_x)$. Then by our initial assumption we have $m_*(\mathcal{O}) \leq \frac{m_*(E)}{\alpha} < \sum_{x=1}^{\infty} m_*(I_x)$. But each I_x is an open set, and so measurable, and so is $\mathcal{O}$, so by theorem 3.3, $m_*(\mathcal{O}) = m(\mathcal{O}) = \sum_{x=1}^{\infty} m(I_x) = \sum_{x=1}^{\infty} m_*(I_x)$. But our assumption $\nexists I_k$ such that $m_*(E \cap I_k) \geq \alpha m_*(I_k)$ implied $m_*(\mathcal{O}) < \sum_{x=1}^{\infty} m_*(I_x)$, a contradiction. Thus our assumption is false and we conclude $\exists I_k$ such that $m_*(E \cap I_k) \geq \alpha m_*(I_k)$.

29. Suppose E contains an interval I. Since it is well ordered, it has a minimum and maxium point, m and M respectively. Then the point 0 is included in the difference interval $M - M = 0$ and it is centered about the origin. $I_{min} = m - M$, $I_{max} = M - m$ and $I_{min} = -I_{max}$. Now suppose E does not contain an interval. Then by 28, $\exists I$ such that $m(E \cap I) \geq \frac{9}{10} m(I)$ For simplicity sake we label $E_0 := E \cap I$. Now suppose the difference set of E_0 does not have an open interval about the origin. Then $\forall x, y \in E_0, |x - y| > \epsilon$, for any ϵ we like. Thus the sets $E_0, E_0 + \epsilon$ are disjoint. By definition, it is clear $(E_0 \cup (E_0 + \epsilon)) \subset (I \cup (I + \epsilon))$. It follows $m(E_0 \cap (E_0 + \epsilon)) \leq m(I \cup (I + \epsilon))$. Since the sets $E_0, E_0 + \epsilon$ are disjoint, it follows $m(2E_0) \leq m(I + 2\epsilon)$. It follows $m(E_0) \leq \frac{m(I)}{2} + \epsilon$. But ϵ was arbitrary, and by assumption $\frac{9}{10} m(I) \leq m(E_0)$. Since our interval I need not have zero measure, we reach a contradiction. Thus the difference set of E_0 has an open interval about the origin. Since $E_0 \subset E$ we conclude E has an open interval about the origin. It is clear that it is

centered at the origin, as any point in this open set about the origin can be written as $z = x - y$, so $\exists z' = y - x = -z$ which concludes our proof.

31. Suppose we are given the constructed set $\mathcal{N}^*$ in exercise 31. Next consider the set of rational numbers over the reals; $\{r_k\}$. Then we claim $[0,1] \subset \cup_{i=1}^{\infty} \mathcal{N} + r_k$. Indeed suppose $x \in [0,1]$. Then by construction $\exists \alpha$ such that $x - x_\alpha = r_k$ since $x - x_\alpha \in \mathcal{E}_\alpha$. Next by monotonicity and ~~sub~~-countable additivity, $1 = m([0,1]) \leq \sum_{i=1}^{\infty} m(\mathcal{N}^*_{r_k}) = \sum_{i=1}^{\infty} m(\mathcal{N}^*)$. Thus $m(\mathcal{N}^*) > 0$. Thus by exercise 29 the difference set of $\mathcal{N}^*$ must have an open interval (a,b) about the origin. But the difference set cannot contain any rationals. This is because if $x - y = r_k | x, y \in \mathcal{N}^*$, then x is equivalent to y and so either x or y is not part of $\mathcal{N}^*$. Since the rationals are dense in R, it follows that any interval centered at zero must contain a rational. Thus there cannot be an open set about the origin for this difference set, which contradicts exercise 29. By assumption for exercise 29, E is measurable and $m(E) > 0$. We showed $m(\mathcal{N}^*) > 0$, so it follows $\mathcal{N}^*$ is not measurable.

1. Notice that this set is simply the same set over $[0,1]$ which is translated to each interval $[n, n+1]$ for integer ns. This is clear when one considers for $|x - \frac{p}{q}| \leq \frac{1}{q^3}$ where $x \in [0,1]$, if we consider an arbitrary set $x \in [n, n+1]$, then $|x + n - \frac{p}{q}| \leq \frac{1}{q^3}$ implies $|x + \frac{nq-p}{q}| \leq \frac{1}{q^3}$ so that if we define $p' = p - nq$ then we have effectively reduced our set over $[n, n+1]$ to the set over $[0,1]$. Now we claim $m(E)_{[0,1]} = 0$. If this claim is true then $\sum m(E)$ over all intervals is 0, and the proof is complete. Thus it is sufficient to show $m(E)_{[0,1]} = 0$. Indeed, let us define $E_k = \{x : |x - \frac{p}{q}| \leq \frac{1}{q^3} | q > k\}$. Then $E_{[0,1]} = \cap_{k=1}^{\infty} \cup_{n \geq k}^{\infty} E_n$. Thus, by the borel-cantilli theorem it is sufficient to show $\sum_{k=1}^{\infty} m(E_k) < \infty$ to prove our claim. Indeed, we first notice that for $p > q$, $|x - \frac{p}{q}| > \frac{1}{q^3}$. Thus we have at most $q+1$ intervals for each $q > k$. It is clear that $\frac{-1}{q^3} \leq x - \frac{p}{q} \leq \frac{1}{q^3}$ so that $m(E_k) \leq (q+1)(\frac{2}{q^3}) = \frac{2}{q^2} + \frac{2}{q^3}$. We also have $\frac{1}{q^3} < \frac{1}{k^3}$ so that $m(E_k) < \frac{2}{k^2} + \frac{2}{k^3}$. Thus $\sum_{k=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} \frac{2}{k^2} + \frac{2}{k^3} < \infty$. Our claim therefore holds and the so $m(E) = 0$.