MATH 354 - Assignment 5

Matthew Stevenson - 260405286

Exercise 2

(b) For $x \in \mathcal{C}$, let $x = \sum_{k=1}^{\infty} a_k 3^{-k} = \sum_{k=1}^{\infty} b_k 3^{-k}$ be two ternary representations of x, where $a_k, b_k \in \{0,2\}$ for all $k \in \mathbb{N}$. Assume $a_k \neq b_k$ for at least one $k \in \mathbb{N}$, else the result is trivial. For these two representations to both be equal to x, WLOG, we must have that $a_k = 0$ for all $k > N$ and $b_k = 2$ for all $k > N$, where $N \in \mathbb{N}$ is some index. Thus:

$$F\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{a^k}{2} = \sum_{k=1}^{\infty} \frac{1}{2^k} c_k \qquad F\left(\sum_{k=1}^{\infty} b_k 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{b^k}{2} = \sum_{k=1}^{\infty} \frac{1}{2^k} d_k$$

Using this result, $c_k = 0$ for all $k > N$ and $d_k = 1$ for all $k > N$; that is, $F\left(\sum_{k=1}^{\infty} a_k 3^{-k}\right) = F\left(\sum_{k=1}^{\infty} b_k 3^{-k}\right)$ are two binary representations of $F(x)$. Therefore, F is indeed well-defined.

Now, take $x \in \mathcal{C}$, then let $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ where $a_k \in \{0,2\}$ for any $k \in \mathbb{N}$. Let $\epsilon > 0$ be given, then there exists some index $N \in \mathbb{N}$ such that for any $n \geq N$, $\epsilon > \frac{1}{2^n}$. Take $y = \sum_{k=1}^{\infty} b_k 3^{-k} \in \mathcal{C}$ very close to x, thus there exists some index $N' \in \mathbb{N}$ such that $a_k = b_k$ for all $k \in \{1, \cdots, N'\}$. Consider then the following:

$$|F(x) - F(y)| = \left|\sum_{k=1}^{\infty} \frac{1}{2^k}\frac{a_k}{2} - \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{b_k}{2}\right| = \left|\sum_{k=N'+1}^{\infty} \frac{1}{2^k}\left(\frac{a_k}{2} - \frac{b_k}{2}\right)\right| \leq \sum_{k=N'+1}^{\infty} \frac{1}{2^k}\left|\frac{a_k}{2} - \frac{b_k}{2}\right| \leq \sum_{k=N'+1}^{\infty} \frac{1}{2^k} < \frac{1}{2^{N'}} < \epsilon$$

as $|\frac{a_k}{2} - \frac{b_k}{2}|$ is greatest (and equal to 1) when either $a_k = 2$ and $b_k = 0$ or $a_k = 0$ and $b_k = 2$. Therefore, F is continuous on $\mathcal{C}$.

Notice that $0 = \sum_{k=1} a_k 3^{-k} \in \mathcal{C}$ where $a_k = 0$ for all $k \in \mathbb{N}$, thus:

$$F(0) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{a^k}{2} = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot 0 = 0$$

Notice that $1 = \sum_{k=1} b_k 3^{-k} \in \mathcal{C}$ where $b_k = 2$ for all $k \in \mathbb{N}$, thus:

$$F(1) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{b^k}{2} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{1 - \frac{1}{2}} - \left(\frac{1}{2}\right)^0 = 2 - 1 = 1$$

□

(c) We can realize each element of the Cantor set as those numbers in $[0,1]$ without any 1s in their ternary decimal expansion. Moreover, any number in $[0,1]$ has a binary decimal representation, so the function F is converting the ternary decimal representation of the Cantor set in terms of 0s and 2s into the binary decimal representation, sending $0 \mapsto 0$ and $2 \mapsto 1$. In such a manner, every possible binary decimal representation is constructed, i.e. the image of F is $[0,1]$. Thus, F is surjective.

□

(d) To show the continuity of F at $x \in [0,1]$, we have to consider two cases: where $x \in \mathcal{C}$ and where $x \in [0,1] - \mathcal{C}$. If $x \in [0,1] - \mathcal{C}$, then F is defined to be in a neighborhood of x Since a constant function is trivially continuous, we have that F is continuous at $x \in [0,1] - \mathcal{C}$.

Now, take $x \in \mathcal{C}$, then let $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ where $a_k \in \{0,2\}$ for any $k \in \mathbb{N}$. Let $\epsilon > 0$ be given, then there exists some index $N \in \mathbb{N}$ such that for any $n \geq N$, $\epsilon > \frac{1}{2^n}$. Take $y = \sum_{k=1}^{\infty} b_k 3^{-k}$ very close to x, thus there exists some index $N' \in \mathbb{N}$ such that $a_k = b_k$ for all $k \in \{1, \cdots, N'\}$. Consider then the following:

$$|F(x) - F(y)| = \left|\sum_{k=1}^{\infty} \frac{1}{2^k}\frac{a_k}{2} - \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{b_k}{2}\right| = \left|\sum_{k=N'+1}^{\infty} \frac{1}{2^k}\left(\frac{a_k}{2} - \frac{b_k}{2}\right)\right| \leq \sum_{k=N'+1}^{\infty} \frac{1}{2^k}\left|\frac{a_k}{2} - \frac{b_k}{2}\right| \leq \sum_{k=N'+1}^{\infty} \frac{1}{2^k} < \frac{1}{2^{N'}} < \epsilon$$

as $|\frac{a_k}{2} - \frac{b_k}{2}|$ is greatest (and equal to 1) when either $a_k = 2$ and $b_k = 0$ or $a_k = 0$ and $b_k = 2$. Therefore, F is continuous both on $\mathcal{C}$ and on $[0,1] - \mathcal{C}$, and is thus continuous on $[0,1]$.

□

Exercise 9

Using the Cantor-set construction algorithm, remove instead the middle quarter of $[0,1]$ and call this set U_1. Remove the middle quarters of each interval in U_1, and call this set U_2. Carry on this process to construct the n^{th} set U_n. These sets look like:

$$U_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$$

$$U_2 = [0, \frac{9}{64}] \cup [\frac{15}{64}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{49}{64}] \cup [\frac{55}{64}, 1]$$

and so on (it's pretty tedious to calculate further as it will be the disjoint union of eight closed intervals...). Let $U = \bigcap_{n=1}^{\infty} U_n$. Since each U_n is a finite union of n closed intervals in $[0,1]$, then U_n is closed. Since U is the countable intersection of closed sets, it is closed. Therefore, $[0,1]-U$ is an open subset of $[0,1]$.

Notice that the measure of the removed portion at the k^{th} iteration $[0,1]-U_k$ is $\left(\frac{1}{2}\right)^{k+1}$. Now, $\overline{[0,1]-U} = \partial U \cup ([0,1]-U)$ and so the boundary of this is the the total removed portion $[0,1]-U$. Thus, we want to compute the Lebesgue measure of $[0,1]-U$, as follows:

$$m(\partial(\overline{[0,1]-U})) = m([0,1]-U) = \sum_{k=1}^{\infty}([0,1]-U_k) = \sum_{k=1}^{\infty}\frac{1}{2^{k+1}} = \sum_{k=0}^{\infty}\frac{1}{2^k} - \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^1 = 2 - \frac{3}{2} = \frac{1}{2}$$

Therefore, $[0,1]-U$ is an open set such that the boundary of its closure has positive Lebesgue measure.

□

Exercise 14

(a) If $E \subset \mathbb{R}$ is closed, then $E = \overline{E}$, so $J_*(E) = J_*(\overline{E})$. If $E \subset \mathbb{R}$ is open, then $E \subset \overline{E}$. Any finite cover of $\overline{E}$ by open intervals is also a finite cover of E by open intervals, so we have monotonicity of the outer Jordan content, i.e. $J_*(E) \leq J_*(\overline{E})$. Let $\{I_j\}_{j=1}^N$ be the finite covering of E by open intervals such that $J_*(E) = \sum_{j=1}^N |I_j|$. For any $\epsilon > 0$, let $I_j' = (I_j - \frac{\epsilon}{2N}) \cup I_j \cup (I_j + \frac{\epsilon}{2N})$, then $|I_j'| = |I_j| + 2\frac{\epsilon}{2N} = |I_j| + \frac{\epsilon}{N}$ and I_j' contains all of the limit points of I_j. Thus, $\{I_j'\}_{j=1}^N$ is a finite covering by open intervals of $\overline{E}$ such that:

$$J_*(\overline{E}) \leq \sum_{j=1}^N |I_j'| = \sum_{j=1}^N \left(|I_j| + \frac{\epsilon}{N}\right) = \left(\sum_{j=1}^N |I_j|\right) + \epsilon = J_*(E) + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we can conclude that $J_*(\overline{E}) \leq J_*(E)$. Therefore, $J_*(E) = J_*(\overline{E})$.

□

(b) Consider $\mathbb{Q} \cap [0,1] \subset [0,1]$. It is also a subset of $\mathbb{Q}$, which is countable, so $\mathbb{Q} \cap [0,1]$ is countable. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\overline{\mathbb{Q} \cap [0,1]} = \mathbb{R} \cap [0,1] = [0,1]$. Thus,

$$J_*(\mathbb{Q} \cap [0,1]) = J_*([0,1]) = 1$$

However, since $\mathbb{Q} \cap [0,1]$ is countable, $m_*(\mathbb{Q} \cap [0,1]) = 0$.

□

Exercise 16

(a) For all $k \in \mathbb{N}$, E_k is measurable. The countable union of measurable sets is measurable, so $\bigcup_{k=n}^{\infty} E_k$ is measurable for all $n \in \mathbb{N}$. Moreover, the countable is intersection of measurable sets is measurable, so $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$ is measurable. Thus, $E = \limsup_{k\to\infty} E_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$ is measurable.

□

(b) Let $B_n = \bigcup_{k=n}^{\infty} E_k$ for all $n \in \mathbb{N}$, then by countable subadditivity, we have that:

$$m(B_n) = m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} m(E_k) < \infty$$

Notice that the B_n's form a non-increasing sequence of sets, so by Corollary 1.3.3, $m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n)$. Moreover, as $\sum_{k=1}^{\infty} m(E_k) < \infty$, for any $\epsilon > 0 \;\exists\; N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} m(E_k) < \epsilon$. Consider now the following:

$$\begin{aligned} m(E) = m\left(\limsup_{k\to\infty} E_k\right) = m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) &= m\left(\bigcap_{n=1}^{\infty} B_n\right) \\ &= \lim_{n\to\infty} m(B_n) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty}\sum_{k=n}^{\infty} m(E_k) \leq \sum_{k=N}^{\infty} m(E_k) < \epsilon \end{aligned}$$

Since $\epsilon < 0$ is arbitrary, we may conclude that $m(E) = 0$.

□

Exercise 21

Recall the non-measurable set $\mathcal{N} \subset [0,1]$ defined in Stein (p. 26) by including one element from each equivalence class of $[0,1]$. These equivalence class are given by $x \sim y$ if $x - y \in \mathbb{Q}$. We have shown in Exercise 2 that the Cantor-Lebesgue function $F\colon \mathcal{C} \to [0,1]$ is surjective, so we can define $E = F^{-1}(\mathcal{N}) \subset \mathcal{C}$, the preimage of $\mathcal{N}$ under F. Moreover, as $m(\mathcal{C}) = 0$ and $E \subset \mathcal{C}$, then E is Lebesgue measurable and $m(E) = 0$. Therefore, we have a continuous function F that maps a Lebesgue measurable set E to a non-measurable set $\mathcal{N}$.

□

Exercise 27
For $t > 0$, consider $\overline{B}(0,t) \cap E_2$, which is a compact subset of E_2 as both E_2 and $\overline{B}(0,t)$ are compact. For $s > t > 0$, consider:

$$|m(\overline{B}(0,s) \cap E_2) - m(\overline{B}(0,t))| = |m(E_2 \cap (\overline{B}(0,s) - \overline{B}(0,t)))| \leq m(\overline{B}(0,s) - \overline{B}(0,t)) \leq C_d \cdot |s-t| \to 0 \text{ as } |s-t| \to 0$$

where C_d is some constant depending on the dimension in E_2 is defined. Therefore, $m(\overline{B}(0,t) \cap E_2)$ is a continuous function in $t \in (0, \infty)$. Since $m(E_1) = a \geq 0$ and $m(E_2) = b \geq 0$, then by the Intermediate Value Theorem, there must exist some $t_0 \in (0, \infty)$ such that $m(\overline{B}(0,t_0) \cap E_2) = c$ for any $c \in (a,b)$. Therefore, $\overline{B}(0,t_0) \cap E_2$ is a compact set such that $E_1 \subset \overline{B}(0,t_0) \subset E_2$ and $m((\overline{B}(0,t_0) \cap E_2) = c$.

□

Exercise 28
Given $\alpha \in (0,1)$, take $\epsilon > 0$ such that $\alpha \cdot m(E) + \epsilon \leq m(E) \implies \alpha \leq 1 - \frac{\epsilon}{m(E)}$, as $m(E) > 0$. By assumption, E is measurable, so there exists an open set $U \subset \mathbb{R}$ such that $E \subset U$ and $m(U - E) < \epsilon$. By monotonicity, $0 < m(E) < m(U)$, so:

$$m(U) - m(E) \leq m(U - E) < \epsilon \implies m(U) - \epsilon < m(E) \implies 1 - \frac{\epsilon}{m(U)} < \frac{m(E)}{m(U)}$$

Moreover, as $0 < m(E) < m(U)$:

$$\frac{1}{m(E)} > \frac{1}{m(U)} \implies -\frac{\epsilon}{m(E)} < -\frac{\epsilon}{m(U)} \implies 1 - \frac{\epsilon}{m(U)} > 1 - \frac{\epsilon}{m(E)} \geq \alpha$$

Now, since $U \subset \mathbb{R}$ is open, there exists a countable family $\{I_j\}_{j=1}^{\infty} \subset \mathbb{R}$ of disjoint open intervals such that $U = \bigcup_{j=1}^{\infty} I_j$. Using the results above along with countable additivity, we have that:

$$m(E) = m(E \cap U) = m(E \cap \left(\bigcup_{j=1}^{\infty} I_j\right)) = m\left(\bigcap_{j=1}^{\infty}(E \cap I_j)\right) = \sum_{j=1}^{\infty} m(E \cap I_j) \geq \alpha \cdot m(U) = \alpha \cdot m\left(\bigcup_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \alpha \cdot m(I_j)$$

Thus, for at least one j, we must have that $m(E \cap I_j) \geq \alpha \cdot m(I_j)$, as required.

□

Exercise 29
As $E \subset \mathbb{R}$ with $m(E) > 0$, by Exercise 28, there exists an open interval $I \subset \mathbb{R}$ such that $m(E \cap I) \geq \frac{9}{10} m(I)$. Denote $E_0 = E \cap I$, and suppose that the difference set $E_0 - E_0$ does not contain an open interval around the origin. For $\epsilon > 0$, take $x \in E_0 \cap (E_0 + \epsilon)$, then $\exists\, y \in E_0$ such that $x = y + \epsilon$, so $x - y = \epsilon \in E_0 - E_0$. Since $\epsilon > 0$ is arbitrary, we must have that $0 \in E_0 - E_0$, which contradicts the fact that $E_0 - E_0$ does not contain an open interval around the origin. Thus, assume that for any $\epsilon > 0$, $E_0 \cap (E_0 + \epsilon) = \emptyset$.

We have that $E_0 \subset I$ and $E_0 + \epsilon \subset I + \epsilon$, so $(E_0 \cup (E_0 + \epsilon)) \subset (I \cup (I + \epsilon))$. By translational invariance of the Lebesgue measure, we know that $m(E_0) = m(E_0 + \epsilon)$. Thus, as $E_0 \cap (E_0 + \epsilon) = \emptyset$:

$$m(E_0 \cup (E_0 + \epsilon)) = m(E_0) + m(E_0 + \epsilon) = 2m(E_0) \geq 2 \cdot \frac{9}{10} m(I) = \frac{9}{5} m(I)$$

However, as $(E_0 \cup (E_0 + \epsilon)) \subset (I \cup (I + \epsilon))$, we have that:

$$m(E_0 \cup (E_0 + \epsilon)) \leq m(I \cup (I + \epsilon)) = m(I) + m((+\epsilon) - I) = m(I) + \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have that $\frac{9}{5} m(I) \leq m(I)$, a contradiction. Thus, E_0 must contain an open interval around the origin. Since $E_0 \subset E$, E must therefore contain an open interval around the origin.

□

Exercise 37
Take an arbitrary compact interval $[a,b] \subset \mathbb{R}$, then f is uniformly continuous on $[a,b]$. Let $\epsilon > 0$ be given, then there exists $\delta > 0$ so that for $x, y \in [a,b]$ such that $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. Construct a partition $\{x_i\}_{i=0}^{n}$ of $[a,b]$, where $x_0 = a$ and $x_n = b$, such that $|x_i - x_{i-1}| < \delta$ for all $i \in \{1, \cdots, n\}$. By the uniform continuity of f on $[a,b]$, for all $i \in \{1, \cdots, n\}$, we have that $|f(x_i) - f_(x_{i-1})| < \frac{\epsilon}{n}$. Notice that:

$$m\left(\Gamma \cap ([a,b] \times f([a,b]))\right) \leq m([a,b] \times f([a,b])) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1}| \cdot |x_i - x_{i-1}| = \sum_{i=1}^{n} \frac{\epsilon}{n} \delta = \epsilon\delta \to 0 \text{ as } \epsilon \to 0$$

Thus, $m\left(\Gamma \cap ([a,b] \times f([a,b]))\right) = 0$ for any compact interval $[a,b] \subset \mathbb{R}$. Notice now that $\bigcup_{n=0}^{\infty} [n-1, n+1] = \mathbb{R}$ and

$f\left(\bigcup_{n=0}^{\infty}[n-1,n+1]\right) = \bigcup_{n=0}^{\infty} f([n-1,n+1]) = \mathbb{R}$, so using our result for an arbitrary compact interval above:

$$\begin{aligned}
m(\Gamma) &= m\left(\Gamma \cap \bigcup_{n=0}^{\infty}([n-1,n+1] \times f([n-1,n+1]))\right) \\
&= m\left(\bigcup_{n=0}^{\infty}(\Gamma \cap ([n-1,n+1] \times f([n-1,n+1])))\right) \\
&\leq m\left(\bigcup_{n=0}^{\infty}[n-1,n+1] \times f([n-1,n+1])\right) \\
&\leq \sum_{n=0}^{\infty} m([n-1,n+1] \times f([n-1,n+1])) \\
&= \sum_{n=0}^{\infty} 0 \\
&= 0
\end{aligned}$$

□

Problem 1

Let $\epsilon > 0$ be given. For $p, q \in \mathbb{Z}$, define the following subset of $\mathbb{R}$:

$$E_{p,q} = \{x \in \mathbb{R} \colon |x - \frac{p}{q}| \leq \frac{1}{q^{2+\epsilon}} \text{ and } \gcd(p,q) = 1\}$$

Let E be the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $\frac{p}{q}$ with relatively prime integers p and q such that $|x - \frac{p}{q}| \leq \frac{1}{q^{2+\epsilon}}$. It is clear that $E = \limsup_{\substack{p\to\infty \\ q\to\infty}} E_{p,q}$. Moreover, notice that:

$$\sum_{p,q=1}^{\infty} m(E_{p,q}) \leq \sum_{p=1}^{\infty}\left(\sum_{\{q\colon\ \gcd(p,q)=1\}} \frac{1}{q^{2+\epsilon}}\right) < \infty$$

The interior sum clearly converges by the p-test, and the exterior sum converges as the tail terms go to zero extremely quickly (proportional to $\frac{1}{q^{2+\epsilon}}$). Furthermore, each $E_{p,q}$ is clearly measurable, so by the Borel-Cantelli Lemma, $m(E) = 0$.

□