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2. The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

(b) The **Cantor-Lebesgue function** is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{if } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } b_k = a_k/2.$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2. Show that F is well defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ as well as $F(1) = 1$.

Solution

Clearly, the ternary series expansion of $x = 0$ is $a_k = 0 \;\forall\; k$, thus, $b_k = 0 \;\forall\; k$ and we have $F(0) = 0$. Taking the expansion in k for $x = 1$, we have $a_k = 2 \;\forall\; k$. Indeed, the sum of the geometric series $\sum_{k=1}^{\infty} 1/3^k = \frac{1}{1-1/3} - 1 = 3/2 - 1 = 1/2$. Thus, taking $a_k = 2 \;\forall\; k$ gives $x = 1$. With this choice of the sequence $a_k, b_k = 1 \;\forall\; k$ and so $F(x) = \sum_{k=1}^{\infty} 1/2^k = 2 - 1 = 1$, thus showing $F(1) = 1$.

Recall that $b_k \in \{0, 1\}$. For continuity on $\mathcal{C}$, let $\varepsilon > 0$ be given and consider $x, y \in \mathcal{C}$. Then $\exists N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$ (why?). Indeed, suppose that the difference is less than ε, there must exist an index N for which all b_k agree, but for which for indices $n > N$, the b_k may differ. The difference between the two series then is at least 2^{-N-2} (if only one term differ) and at most 2^{-N} (if all terms for index greater than N differ).

Thus, for all $n < N$, $b_{k_x} = b_{k_y}$. Thus,

$$\begin{aligned}
|x - y| &= \left| \sum_{k=1}^{\infty} 2\frac{b_{k_x}}{3^k} - 2\frac{b_{k_y}}{3^k} \right| \\
&= \left| 2\sum_{k=1}^{N} \frac{b_{k_x} - b_{k_y}}{3^k} + 2\sum_{k=N+1}^{\infty} \frac{b_{k_x} - b_{k_y}}{3^k} \right| \\
&\leq 2\sum_{k=N+1}^{\infty} \frac{1}{3^k} \\
&= \frac{3}{2} - \frac{3}{2}\left(1 - \frac{1}{3^{N+1}}\right) = \frac{1}{3^N}
\end{aligned}$$

thus take $\delta = 3^{-(N+1)}$. For this choice, we have indeed

$$\begin{aligned}|F(x)-F(y)| &= \left|\sum_{k=1}^{\infty}\frac{b_{k_x}}{2^k}-\frac{b_{k_y}}{2^k}\right| \\ &\leq \sum_{k=N+1}^{\infty}\frac{1}{2^k} \\ &= 2^{N+1} < \varepsilon.\end{aligned}$$

To show now that $F(x)$ is well defined, we want if $x = y$, then $F(x) = F(y)$. First, we claim that this entail that the expansion $a_{k_x} = a_{k_y}$ $\forall$ k. For, suppose not and let m be any natural number for which they disagree and wlog, say $a_{m_x} > a_{m_y}$. Then, in the m^{th} step of the construction of the Cantor set, we get x is in the left endpoint of a right subinterval, y is the left endpoint of the left subinterval which are at distance 2^{-m} apart. Thus $x \neq y$, which is a contradiction. Since all terms in the series expansion are equal, so $a_{k_y} = a_{k_x}$, then $b_{k_y} = b_{k_x}$ $\forall$ k and we conclude that $F(x) = F(y)$.

(c) Prove that $F : \mathcal{C} \to [0,1]$ is surjective, that is, for every $y \in [0,1]$ there exists $x \in \mathcal{C}$ such that $F(x) = y$.

Let $y \in [0,1]$. Then, there exists a unique binary expansion such that $y = \sum_{k=1}^{\infty} b_{k_y}/2^k$. Taking $a_{k_y} = b_{k_y} \cdot 2$, we get a ternary series expansion of the form $\sum_{k=1}^{\infty} a_{k_y}3^{-k}$. It remains thus to show that this element, say x, is in $\mathcal{C}$. We proceed by induction on the points created at the n^{th} step of the construction of the Cantor set. Clearly, the left endpoints in the first step of the construction are 0 and $2/3$. Suppose that we have at step $\mathcal{C}_n$ removed the intervals and by the induction hypothesis all left endpoints have ternary expansion with $a_k \in \{0,2\}$, where $x = \sum_{k=1}^{n} a_k 3^{-k}$. Thus, for any x satisfying the above conditions, in $\mathcal{C}_{n+1}$, for in the interval the expansion in 0 corresponding to the left endpoint of the left subinterval (equal to x) or the left endpoint of the right subinterval (corresponding to the the left endpoint to the right of the term at distance d_m, which corresponds to $2/3^{-m}$ for m the first natural number less than n for which the expansion differ by 2. Namely, we have

$$\sum_{k=1}^{n}\frac{a_k}{3^k} \leq x \leq \sum_{k=1}^{n}\frac{a_k}{3^k} + \sum_{k=N+1}^{\infty}\frac{2}{3^{n+1}} = \sum_{k=1}^{n}\frac{a_k}{3^k} + \frac{1}{3^n}$$

Since no endpoint is removed in the Cantor set, all points in which $a_k = \{0,2\}$ are in $\mathcal{C}_n$, and this for all n.

(d) One can also extend F to be a continuous function on $[0,1]$ as follows. Note that if (a,b) is an open interval of the complement of $\mathcal{C}$, then $F(a) = F(b)$. Hence we may define F to have the constant value $F(a)$ in that interval.

Solution
F is the Devil's staircase, and is right-continuous. Let $x, y \in [0,1]$ be given. If x, y are as in (b), we have continuity. Given a step height ε, we want to find the distance from which two points are away from one another by less than δ imply that the steps are less than ε . Let ε be given, there exists N such that $\varepsilon < 2^{-N}$. Using the same continuity on the Cantor set, we can take $\delta = 3^{-N-2}$. I claim this δ works. Indeed, if $|x-y| < \delta$, then there exists $c_1, c_2 \in \mathcal{C}$ (not necessarily different from x, y) with $F(c_1) = F(x)$ and $F(c_2) = y + 2^{N-1}$. There are multiple cases now: if x, y are in $\mathcal{C}$, we are good. Similarly, if one of the endpoints is in the Cantor set, but y is not, $\exists c_2 \in \mathcal{C}$ which is the next endpoint on the right of y. By the choice of δ, using the triangle inequality, we have $|x-y| = |x - c_2 + c_2 - y| \leq |x - c_2| + |c_2 - y| \leq 2/3^{-N-2} < 3^{-N-1}$. Now, by the triangle inequality,

$$|F(x) - F(y)| = |F(x)F(c_2) + F(c_2) - F(y)| < 2^{-N-1} + 2^{-N-1} = 2^{-N}.$$

More generally, take x to be a point not in the Cantor set, then $\exists c_1 \in \mathcal{C}$ with $c_1 < x$, yet $F(c_1) = F(x)$. Therefore, using the same argument as above

$$\begin{aligned}
|x-y| &= |x - c_1 + c_1 - c_2 + c_2 - y| \\
&\leq |x - c_1| + |c_1 - c_2| + |y - c_2| \\
&< \frac{3}{3^{N+2}} = \frac{1}{3^{N+1}}
\end{aligned}$$

then from the continuity in (b) and the construction of the Cantor set intervals,

$$\begin{aligned}
|F(x) - F(y)| &= |F(x) - F(c_1) + F(c_1) - F(c_2) + F(c_2) - F(y)| \\
&\leq |F(c_1) - F(c_2)| \\
&\leq \varepsilon = \frac{1}{2^N}
\end{aligned}$$

by construction and the choice of δ. This proves that F extends to a continuous function on $[0,1]$.

9. **Extra-credit.** Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.
[Hint: Consider the set obtained by taking the union of open intervals which are deleted at the odd steps in the construction of a Cantor-like set.]

Solution
The fat Cantor set here is such example. Consider a set in which we remove (instead of the third intervals remove from [0,1]) the subintervals of length 2^{-2n} from the middle of each of the 2^{n-1} intervals at step $n \in \mathbb{N}$. Thus, the intervals of the set $\mathcal{C}_f$, constructed iteratively left are precisely at step one: $\mathcal{C}_{f1} = [0, 3/8] \cup [5/8, 1]$, etc. The measure is

then the sum of the measure of each closed interval, since the countable union of those closed intervals is closed and we can take the infimum over the cover by closed intervals to be precisely this, then

$$\begin{aligned} m(\mathcal{C}_f) &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \\ &= \frac{1}{4} \sum_{m=0}^{\infty} \frac{1}{2^m} \\ &= \frac{1}{4}\left(1 - \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

so the measure of $\mathcal{C}_f$ is zero. Thus, by construction $\mathcal{C}_f$ is closed, being the countable union of closed sets, so the subspace restriction $[0,1] \setminus \mathcal{C}_f$ is open and has measure $1/2$.

We can construct an sequence of limit points that converge to any points in $[0,1]$, constructed by taking a point in $S \equiv [0,1] \setminus \mathcal{C}_f$ that converge to some point in $\mathcal{C}_f$. The closure of the set S is $[0,1]$ and since $\mathcal{C}_f^\circ \cup \partial\mathcal{C}_f = \overline{\mathcal{C}_f}$, and since the boundary must contain at least $\mathcal{C}_f$, then we infer that $\partial\mathcal{O} = \partial[0,1] \setminus \mathcal{C}_f$ has measure at least one half.

14. The purpose of this exercise is to show that covering by a finite number of intervals will not suffice in the definition of the outer measure m_*.
The **outer Jordan content** $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|$$

where the inf is taken over every *finite* covering $E \subset \bigcup_{j=1}^{N} I_j$, by intervals I_j.

(a) Prove that $J_*(E) = J_*(\overline{E})$ for every set E (here $\overline{E}$ denotes the closure of E).
Proof. Since by definition $E \subseteq \overline{E}$, we must have for a finite cover $\bigcup_{j=1}^{N_c} I_{j_c}$ of $\overline{E}$ that $E \subseteq \bigcup_{j=1}^{N_c} I_{j_c}$, thus since we are taking the infimum for E over a bigger set, $J_*(E) \leq J_*(\overline{E})$.
Conversely, consider an arbitrary finite cover of E by intervals. For each interval I_j, take C_j to be the infimum over all closed interval containing I_j (here we assume $I_j \neq \emptyset$), so that $|C_j| \leq (1+\varepsilon)|I_j|$. This corresponds to adding the endpoints of each I_j Taking the infimum over all finite covers, we get that C_j are almost disjoint

(otherwise, the I_j overlap and one could choose non-overlapping intervals I_i, such that $\sum_{i=1}^N I_i < \sum_{j=1}^N I_j$, which is a contradiction). Also, since $E \subseteq \bigcup_{j=1}^N Q_j, \overline{E} \subseteq \overline{\bigcup_{j=1}^N Q_j}$, but this infimum must be equal to $\bigcup_{j=1}^N \overline{Q_j}$ since the finite union of closed sets is closed. Thus, by countable sub-additivity, we have

$$\bigcup_{j=1}^N |C_j| \leq \sum_{j=1}^N |C_j| \leq \sum_{j=1}^N |I_j|(1+\varepsilon)$$

where ε is chosen arbitrarily small. ■

(b) Exhibit a countable subset $E \in [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

Solution

Our good old friend $\mathbb{Q} \cap [0,1]$. Indeed, by (a), we know that $J_*(\mathbb{Q} \cap [\prime, \infty]) = J_*(\overline{\mathbb{Q} \cap [0,1]}) = J_*(\mathbb{R} \cap [0,1])$ which is just 1 since we can cover by the closed interval [0,1], of length 1. The outer measure of $\mathbb{Q} \cap [0,1]$ is however 0, since we can cover each rational in the interval by a countable number of degenerate boxes (consisting of the rational point). Let n be the position of rational $q \in [0,1]$ using Cantor diagonalization argument. Using countable sub-additivity, we are summing a countable number of measure zero set, so $\sum_{i=1}^n m_*(q_n)$=0. if one wants, cover them instead with boxes of length $\varepsilon/2^n$ so that $q_n \in \mathbb{Q} \subset Q_n = [q_n - \varepsilon/2^n, q_n + \varepsilon/2^n]$ as above. Then by countable sub-additivity,

$$\begin{aligned} m_*\left(\bigcup_{n=1}^\infty Q_n\right) &\leq \sum_{n=1}^\infty m_*(Q_n) \\ &\leq \sum_{n=1}^\infty \frac{2\varepsilon}{2^n} = 2\varepsilon. \end{aligned}$$

and since ε was arbitrary, we conclude that the measure of a countable set is zero. Thus, $m_*(\mathbb{Q} \cap [0,1]) = 0$, while $J_*(\mathbb{Q} \cap [0,1]) = 1$.

16. **The Borel-Cantelli lemma.** Suppose $\{E_k\}_{k=1}^\infty$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^\infty m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k\to\infty}(E_k) \end{aligned}$$

(a) Show that E is measurable.

Solution
Since E_k is a family of measurable subsets and that countable union of measurable sets is measurable, then for each $k \geq 1$, select $\mathcal{O}_k$ open such that $m_*(\mathcal{O}_k - E_k) < \varepsilon/2^k$. Write $B_n = \bigcup_{k=n}^{\infty} E_k$. The countable union of $\mathcal{O}_k$, $\mathcal{O}_n$ is open and by sub-additivity,

$$m_*(\mathcal{O}_n - B_n) < \sum_{k=n}^{\infty} m_*(\mathcal{O}_k - E_k) \leq \varepsilon.$$

thus each B_n is measurable. Since complement of measurable sets are measurable, we have therefore that the countable intersection is. These properties are proved on p.18 of the book.

(b) Prove $m(E) = 0$.
[Hint: Write $E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$.]

Solution
First, at least one of E_k is finite, since $\sum_{k=1}^{\infty} m(E_k) < \infty$. Let B_n as in the previous part, B_n forms a decreasing sequence of sets, since we take the union over less sets, thus $B_n \supseteq B_{n+1} \supseteq \cdots$ for all n. Using continuity from above, and the fact that $m(E_k) < \infty$ for some k, then by continuity from above, we have

$$\lim_{n\to\infty} m(B_n) = m\left(\bigcap_{n=1}^{\infty} B_n\right).$$

thus

$$m\left(\limsup_{k\to\infty} E_k\right) = m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right)$$

and from countable sub-additivity, we have furthermore that

$$m\left(\bigcup_{n=1}^{\infty} B_n\right) \leq m\left(\sum_{n=1}^{\infty} B_n\right).$$

Since the sum on the right hand side converges, $\forall\, \varepsilon > 0$, $\exists N > n, N \in \mathbb{N}$ such that for $m_1, m_2 > N$, $m(E_{m_1}) - m(E_{m_2}) < \varepsilon$ (we can bound using Cauchy sequences the partial sum for the tail). Fix m_1; passing to the limit (letting $m_2 \to \infty$), we see that the right hand side goes to zero, thus $m(E) = 0$.

21. Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.
[Hint: Consider a non-measurable subset of $[0, 1]$, and its inverse image in $\mathcal{C}$ by the function F in Exercise 2.]

Solution

Indeed, consider the non-measurable set $\mathcal{N}$ constructed in class using the axiom of choice. Using the hint, we consider the function $F : \mathcal{C} \to [0,1]$, which is surjective and continuous by 2. Consider then the pre-image set $F^{-1}(\mathcal{N}) \equiv \{x \in \mathcal{C}, F(x) \in \mathcal{N}\}$. Taking $f : F^{-1}(\mathcal{N}) \to \mathcal{N}$ and the function f to be the restriction of $F(x)|_{x \in \mathcal{N}}$, this function is also surjective and continuous since $F(x)$ was. Now, since $F^{-1}(\mathcal{N}) \subset \mathcal{C}$, by monotonicity $0 \leq m(F^{-1}(\mathcal{N})) \leq m(\mathcal{C}) = 0$, so $F^{-1}(\mathcal{N})$ is measurable, while $f(F^{-1}(\mathcal{N})) = \mathcal{N}$ is not measurable.

27. **Extra credit.** Suppose E_1 and E_2 are a pair of compact sets in $\mathbb{R}^d$ with $E_1 \subset E_2$, and let $a = m(E_1)$ and $b = m(E_2)$. Prove that for any c with $a < c < b$, there is a compact set E with $E_1 \subset E \subset E_2$ and $m(E) = c$.
[Hint: As an example, if $d = 1$ and E is a measurable subset of $[0,1]$, consider $m(E \cap [0,t])$ as a function of t.]

Solution

Consider the function $f : m(E_1 \cup E_2 \cap C_t)$, $f : [a,b] \to \mathbb{R}$, where we take the cube C_t centered at zero with side length of $t \in [0,l]$ such that C_l, is a hypercube of side length l in which E_2 is properly contained, thus $C_l \supset E_2$. Then, taking cubes C_t of measure thus measure $m(C_t) = t^d$. Since E_1, E_2 are bounded, we can consider $E_1 \subset C_{E_1}$ and $E_2 \subset C_{E_2}$ since compact in $\mathbb{R}^d$ is equivalent to closed and bounded. If $t = 0$, then $E_2 \cap C_0 = \emptyset$, and so $f(0) = m(E_1) = a$. Similarly, if $t = l$, then by construction $C_{E_2} \cap E_2 = E_2$ and $f(E_2) = m(E_2 \cap E_1) = m(E_2) = b$ since $E_1 \subset E_2$. Clearly, since the intersection and union of closed sets is again closed, and since $E \subset E_2$ bounded, E must be compact. Furthermore, the function f defined above is continuous. Indeed, f is monotone increasing for t and we have for ε given, then we know that for $x, y \in [0,l]$,

$$\begin{aligned}
&|m(E_1 \cup (E_2 \cap C_x)) - m(E_1 \cup (E_2 \cap C_y))| \\
&= |m(E_1^{\complement} \cap E_2 \cap C_x) - m(E_1^{\complement} \cap E_2 \cap C_y)| \\
&\leq |m(C_x) - m(C_y)| \\
&= |x^d - y^d|
\end{aligned}$$

and in particular, if x or wlog $y \neq 0$, then we can write $a = x/y$ and the above will be $y^d(a^d - 1) = y^d(a-1)(a^{d-1} + \cdots + 1)$ and taking x, y close enough, we get $a - 1 = y|x-y|$, thus taking $|x - y| < \delta$, we get continuity.

By the Intermediate value theorem, since the function is continuous on $[0,1]$ and $a < c < b$, with $E_1 \subset E \subset E_2$, then $\exists t_0 \in [0,1]$ such that $f(t_0) = c$. Taking this delta, we get that $E \equiv E_1 \cup E_2 \cap C_{t_0}$ which is compact corresponds to the desired result.

28. Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$ there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha m_*(I).$$

Loosely speaking, this estimate shows that E contains almost a whole interval.
[Hint: Choose an open set $\mathcal{O}$ that contains E, and such that $m_*(E) \geq \alpha m_*(\mathcal{O})$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.]

Solution
Here, use the fact that every open set in $\mathbb{R}$ can be decomposed uniquely into a countable union of disjoint sets, that is can write $O \subset \mathbb{R}$, $O = \bigsqcup_{j=1}^{\infty} I_j$.

The case where $m_*(E) = \infty$ will be dealt with later. Assume that the outer measure of the set E is non-zero and finite. Let $\varepsilon > 0$ given be such that $\alpha m_*(E) + \varepsilon \leq m_*(E)$. Indeed, from the definition of infimum, ans assuming $m_*(E) > 0$, then $m_*(E)(1-\alpha) > 0$, so $\exists \varepsilon > 0$ such that $m_*(E)(1-\alpha) > \varepsilon$, yielding the above claim. We thus have

$$\alpha \leq 1 - \frac{\varepsilon}{m_*(E)}$$

By regularity, $\exists \mathcal{O}$ open such that $m_*(\mathcal{O} \setminus E) < \varepsilon$, $E \subset \mathcal{O}$ so that $m_*(\mathcal{O}) < m_*(E) + \varepsilon$. Now, since $m_*(\mathcal{O}) > m_*(E) > 0$ by monotonicity, we can safely divide through by $m_*(\mathcal{O})$ to get

$$1 - \frac{\varepsilon}{m_*(\mathcal{O})} < \frac{m_*(E)}{m_*(\mathcal{O})} \Rightarrow 1 - \frac{\varepsilon}{m_*(\mathcal{O})} > 1 - \frac{\varepsilon}{m_*(\mathcal{O})} \geq \alpha$$

and from above, we get $m_*(E)/m_*(\mathcal{O}) \geq \alpha$, therefore $m_*(E) \geq \alpha m_*(\mathcal{O})$. We now use the decomposition of $\mathcal{O}$ into disjoint intervals and write

$$\begin{aligned} m_*(E) = m_*(E \cap \mathcal{O}) &= m_* \left(E \cap \bigsqcup_{j=1}^{\infty} I_j \right) \\ &= m_* \left(\bigsqcup_{j=1}^{\infty} E \cap I_j \right) \\ &\leq \sum_{j=1}^{\infty} m(E \cap I_j) \end{aligned}$$

where the last two steps use De Morgan's law and countable additivity (the sets $E \cap I_j$ being disjoint). To conclude, since we have disjoint I_j, it follows that

$$\sum_{j=1}^{\infty} m(E \cap I_j) \geq \alpha m_*(\mathcal{O}) = \sum_{j=1}^{\infty} \alpha m_*(I_j)$$

and this inequality entails that for at least one j, say j_0, we have $m_*(E \cap I_{j_0}) \geq \alpha m_*(I_{j_0})$.

Now, if $m_*(E)$ is infinite, then we can consider the open interval $\mathbb{R}$. Then, we have $E \subset \mathbb{R}$, so the LHS is $\infty = m_*(E) \geq \alpha m_*(\mathbb{R}) = \infty$, which is degenerate, but works.

29. Suppose E is a measurable subset of R with $m(E) > 0$. Prove that the difference set of E, which is defined by

$$z \in \mathbb{R} : z = x - y \text{ for some } x, y \in E,$$

contains an open interval centered at the origin. If E contains an interval, then the conclusion is straightforward. In general, one may rely on Exercise 28.
[Hint: Indeed, by Exercise 28, there exists an open interval I so that $m(E \cap I) \geq (9/10)m(I)$. If we denote $E \cap I$ by E_0, and suppose that the difference set of E_0 does not contain an open interval around the origin, then for arbitrarily small a the sets E_0, and $E_0 + a$ are disjoint. From the fact that $(E_0 \cup (E_0 + a)) \subset (I \cup (I + a))$ we get a contradiction, since the left-hand side has measure $2m(E_0)$, while the right-hand side has measure only slightly larger than $m(I)$.]

Solution
Clearly, since E is non-empty (it has positive measure), $\exists x \in E$ and $z = 0 = x - x \in E - E \equiv Z$. If E is an nonempty interval, then there exists $\sup E = b_z$ and $\inf E = a_z$ and the set Z contains the open interval $(-|b_z - a_z|, |b_z - a_z|)$ around zero, possibly excluding the endpoints.

If E is not an interval (or doesn't contain one), then we may rely as on Ex. 28, using the fact that there exists I open such that $m(E \cap I) \geq \frac{9}{10}m(I)$. Let $E \cap I \equiv E_0$, then $\exists a = \min\{\inf_{x,y \in E} |x - y|, \frac{4}{5}m(I)\} > 0$, $a > 0$ chosen arbitrarily small. Furthermore, $\exists \varepsilon > 0$ such that $a - \varepsilon > 0$. Finding such a is possible due the positive measure of $E \subset I$ (cannot contain a single point, or only countably many isolated points, and we are assuming that it doesn't contain an interval, thus the distance between two points is positive). From this, it follows that E_0 and $E_0 - a$ are disjoint (see assignment 1 or Observation 4 in Stein & Shakarchi). Here, $E_0 + a = \{x + a : x \in E_0\}$

Since $d(E_0, E_0 + a) > 0$ are at distance $a - \varepsilon > 0$, then using countable sub-additivity and the fact that the sets are disjoint, we have

$$m(E_0 \cup (E_0 + a)) = m(E_0) + m(E_0 + a)$$

both of which are measurable since I is measurable, and hence $E_0 = E \cap I$ is also measurable. The Lebesgue measure coincides with the outer measure and both are translation invariant. Now $E_0 = E \cap I \subset I$, and also $E_0 + a \subset I + a$, thus both E_0 and $E_0 + a$ are contained in $I \cup (I + a)$. We have by monotonicity, by translation invariance and countable sub-additivity that

$$2m(E_0) = m(E_0 \sqcup (E_0 + a)) \leq m(I \cup (I + a))$$

where $I \cup (I + a)$ is an interval by the choice of a, which is $(\inf(I), \sup(I) + a)$. We can write any open subset in $\mathbb{R}$ as the disjoint union of countably many open intervals, so

$I \cup (I+a) = I + J$, where J is the interval $(\sup(I), \sup(I)+a)$ of Lebesgue measure a. By countable sub-additivity, $m(I \cup (I+a)) = m(I \sqcup J) = m(I) + m(J) \leq m(I) + \frac{4}{5}m(I)$. But then

$$\frac{9}{5}m(I) > m(I) + a \geq 2m(E_0) \geq \frac{9}{5}m(I)$$

which is a contradiction. Thus, we conclude that there exists an open interval I centered around the origin.

31. **Extra credit.** The result in Exercise 29 provides an alternate proof of the non-measurability of the set $\mathcal{N}$ studied in the text. In fact, we may also prove the non-measurability of a set in $\mathbb{R}$ that is very closely related to the set $\mathcal{N}$.
Given two real numbers x and y, we shall write as before that $x \sim y$ whenever the difference $x - y$ is rational. Let $\mathcal{N}^*$ denote a set that consists of one element in each equivalence class of $\sim$. Prove that $\mathcal{N}^*$ is non-measurable by using the result in Exercise 29.
[Hint: If $\mathcal{N}^*$ is measurable, then so are its translates $\mathcal{N}_n^* = \mathcal{N}^* + r_n$, where $\{r_n\}_{n=1}^{\infty}$ is an enumeration of $\mathbb{Q}$. How does this imply that $m(\mathcal{N}^*) > 0$? Can the difference set of $\mathcal{N}^*$ contain an open interval centered at the origin?]

Solution
We proceed by contradiction, assuming that $\mathcal{N}^*$ is measurable. If we consider as indicated in the hint the set $\mathcal{N}_N^*$, which is the translation of the set $\mathcal{N}^*$ by r_n, then since by construction our equivalence classes are real numbers that differ by a rational, we can enumerating all of $\mathbb{Q}$ take the countable union of these translate, which have same measure. Now, if we take

$$m\left(\bigcup_{n=1}^{\infty} \mathcal{N}^* + r_n\right) = m(\mathbb{R}) = \infty,$$

and by countable sub-additivity, we know that

$$m\left(\bigcup_{n=1}^{\infty} \mathcal{N}^* + r_n\right) \leq \sum_{n=1}^{\infty} m(\mathcal{N}^* + r_n) = \sum_{i=1}^{\infty} m(C_n^*) = \infty$$

from the previous result. On the other hand, we have if $\mathcal{N}^*$ is measurable that for sets of positive measure, then the difference set of $\mathcal{N}^*$ contains an open interval centered at the origin. But this cannot happen, since then $x - y = 0$ would imply $x \sim y$, and since the set consists of the selection of one element (using the axiom of choice) from each equivalent class, then 0 is not in the difference set; we infer that $m(\mathcal{N}^*) = 0$, so together this show that actually, $\mathcal{N}^*$ is not measurable.

37. **Extra credit.** Suppose Γ is a curve $y = f(x)$ in $\mathbb{R}^2$, where f is continuous. Show that $m(\Gamma) = 0$.
[Hint: Cover Γ by rectangles, using the uniform continuity of f.]

Solution
Consider an interval $E_n[n, n+1]$ for $n \in \mathbb{Z}$. Then, for all n, the union of these sets form a partition of $\mathbb{R}$ of almost disjoint closed intervals. For any such interval, we have since the function is continuous absolute continuity for free on this interval. Let $\varepsilon = 2^{-k}$, then we can tile the unit interval with N different rectangles. By absolute continuity, we have that there exists δ_ε for each given ε which corresponds here to the base of each rectangle. Taking $\delta = \min\{1/N, \delta_\varepsilon\}$ given, then the total area of the cover for $\Gamma|_{[n,n+1]}$ is $N \times 4\delta/2^k \leq 4/2^k$. This holds for all k, thus from this we infer that $m(\Gamma|[n, n+1]) = 0$, taking $k \to \infty$. Since our choice of partition of the real line is countable, we can then using countable subadditivity to take

$$m(\Gamma) = \bigcup_{n\in\mathbb{Z}} m(\Gamma \cap E_n) \leq \sum_{n\in\mathbb{Z}} m(\Gamma \cap E_n) = 0$$

since each individual $\Gamma \cap E_n$ has measure zero for $n \in \mathbb{Z}$. Therefore, $m(\Gamma) = 0$.

we alternatively can take interval $E=(-n,n)$ with $\varepsilon = \frac{1}{n 2^{k+1}}$ prove that $m(E_n)=0$ on this interval using the above and take $\lim_{n\to\infty}$ to get the result

1. Given an irrational x, one can show (using the pigeon-hole principle, for example) that there exists infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

However, prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3} \quad (\text{or } \leq 1/q^{2+\epsilon})$$

is a set of measure zero.
[Hint: Use the Borel-Cantelli lemma.]

Solution
Let x be given and consider the interval $[n-1, n]$ for $n \in N$ in which x lie. Wlog, suppose x is positive. We want to consider the set of x such that

$$-\frac{1}{q^2} \leq x - \frac{p}{q} \leq \frac{1}{q^2}$$

and

$$E_n = \left\{ x_n \in [n-1, n) : \exists \text{ infinitely many } p, q \text{ coprime for which } \left| x_n - \frac{p}{q} \right| \leq \frac{1}{q^3} \right\}$$

Let

$$E_q \equiv \bigcup_{\substack{p \\ (n-1)q \leq p \leq nq \\ p,q,\text{coprime}}} \left(\frac{p}{q} - \frac{1}{q^3}, \frac{1}{q^3} + \frac{p}{q}\right)$$

where q is fixed. Now,

$$m(E_q) \leq 2 \sum_{\substack{p \\ (n-1)q \leq p \leq nq \\ p,q,\text{coprime}}} m\left(\frac{1}{q^3}\right) \leq 2m\left(\frac{nq+n}{q^3}\right) \leq 2m\left(\frac{np+1}{q^3}\right) \leq \frac{4n}{q^2}$$

Thus, we have $m\left(\bigcup_{q=1}^{\infty} E_q\right) \leq \sum_{q=1}^{\infty} m(E_q) \leq \sum_{q=1}^{\infty} \frac{4n}{q^2} < \infty$ since the latter series converge by the p test.

Now, we can invoke the Borel-Cantelli lemma. Since

$$\sum_{q=1}^{\infty} m(E_q) < \infty \quad \Rightarrow \quad m\left(\limsup_{q \to \infty} E_q\right) = 0$$

by the previous problem. Considering the countable union of the sets

$$E = \bigcup_{n \in \mathbb{N}} E_n : \{x_n \in \mathbb{Q}^{\complement}, (n-1) \leq x \leq n, n \in \mathbb{N}\}$$

over $\mathbb{N}$, since each set is almost disjoint, then by countable sub-additivity, the corresponding sets have measure less than the sum, and since each set as measure zero,

$$0 \leq m(E) = m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n) = 0$$

so $m(E) = 0$. It is straightforward to extend this by adjoining the sets of negative irrationals satisfying the above conditions. Label them E_{-n+1} for $-x_n$ described above, where $x_n > 0 \in \mathbb{Q}^{\complement}$, to E. Since $\mathbb{Z}$ is countable, a similar argument holds for

$$E_{-n} = \left\{x_n > 0 \in \mathbb{Q}^{\complement} : \left|-x_n - \frac{-p}{q}\right| \leq \frac{1}{q^3}\right\}$$

and the union of these two sets is again less than the measure of the sum, which are both zero. This is obvious (symmetry).