McGill University
Math 354: Honors Analysis 3

Practice problems — not for credit

Problem 1. Determine whether the family of $\mathcal{F} = \{f_n\}$ functions $f_n(x) = x^n$ is uniformly equicontinuous.
1st Solution: The family $\mathcal{F}$ is clearly uniformyl bounded. If it were uniformyl equicontinuous, we could apply Arzela-Ascoli's theorem to conclude that a sequence f_n has a subsequence that converges uniformly in $C[0,1]$; the limit would have to be a continuous function $g(x)$. However, it is easy to see that $f_n(x) \to h(x)$ as $n \to \infty$, where

$$h(x) = \begin{cases} 0, & x \in [0,1), \\ 1, & x = 1. \end{cases}$$

This function has a jump discontinuity at $x = 1$ and so the convergence cannot be uniform, hence $\mathcal{F}$ is not uniformly equicontinuous.
2nd Solution: Alternatively, we can show that for any sequence n_k, the sequence of functions f_{n_k} cannot be a Cauchy sequence in $C[0,1]$ (which would be necessary for uniform convergence). Indeed, fix some $m = n_k$, and consider the d_∞ distance between x^m and $x^n, n = n_{k+1}, n_{k+2}, \ldots$ We claim that $\limsup_{n\to\infty} d_\infty(x^m, x^n) \geq 1/2$.

Indeed, choose x_0 s.t. $x_0^m > 3/4$, say. Then let N be such that $x_0^n < 1/4$ for $n > N$. Then for any $n_k > N$, we have $x_0^m - x_0^{n_k} \geq 3/4 - 1/4 = 1/2$, hence the same inequality holds for d_∞, QED.
3rd Solution: Finally, take $x = 1$ in the definition of the uniform equicontinuity. Then for any *fixed* $\delta > 0$, we clearly have $\lim_{n\to\infty}(1-\delta)^n = 0 \neq 1 = f_n(1)$, which shows that for large enough n, $|f_n(1-\delta) - f_n(1)| \geq 1/2$, and so the family $\mathcal{F}$ cannot be uniformly equicontinuous at $x = 1$.

Problem 2. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is differentiable for each $n \in \mathbb{N}$. Suppose also that $\{f_n'\}$ converges uniformly on $\mathbb{R}$ and that $\{f_n(0)\}$ converges. Then $\{f_n\}$ converges pointwise on $\mathbb{R}$.
Solution. Fix $x \in \mathbb{R}, x \neq 0$, and let $\epsilon > 0$ be given. Since $\{f_n(0)\}$ converges it is Cauchy, so we may choose $N_1 \in \mathbb{N}$ so that

$$m, n \geq N_1 \text{ implies } |f_n(0) - f_m(0)| < \frac{\epsilon}{2}$$

Furthermore, since $\{f_n'\}$ is uniformly convergent it is uniformly Cauchy. Therefore we may choose $N_2 \in \mathbb{N}$ so that

$$m, n \geq N_2 \text{ implies } |f_n'(y) - f_m'(y)| < \frac{\epsilon}{2|x|} \text{ for all } y \in \mathbb{R}$$

Now set $N = \max(N_1, N_2)$ and fix any $m, n \geq N$. Then use the mean value theorem to choose $c \in (0, x)$ so that

$$(f_n - f_m)'(c)(x - 0) = (f_n - f_m)(x) - (f_n - f_m)(0)$$

Then using the above inequalities we find that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(0) - f_m(0)| + |f_n(x) - f_n(0) + f_m(0) - f_m(x)| \\ &= |f_n(0) - f_m(0)| + |(f_n - f_m)(x) - (f_n - f_m)(0)| \\ &= |f_n(0) - f_m(0)| + |(f_n - f_m)'(c)(x - 0)| \\ &= |f_n(0) - f_m(0)| + |f_n'(c) - f_m'(c)||x| \\ &< \epsilon \end{aligned}$$

We conclude that $\{f_n(x)\}$ is Cauchy and hence convergent. Thus $\{f_n\}$ converges pointwise.

Problem 3. Suppose that $A \subset \mathbb{R}^n$ and that $F : A \to \mathbb{R}^m$ is continuous. If A is path connected, then its graph

$$G = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n+m} | \mathbf{u} \in A, \mathbf{v} = F(\mathbf{u})\}$$

is also path connected.

Solution. Let $(\mathbf{u}_1, F(\mathbf{u}_1))$ and $(\mathbf{u}_2, F(\mathbf{u}_2))$ be two points in G. Since A is path connected, there exists a continuous path $\phi : [0,1] \to A$ connecting $\mathbf{u}_1$ and $\mathbf{u}_2$. Consider the path $\Gamma : t \to (\gamma(t), F(\gamma(t))) \in G$. The function $F(\gamma(t))$ is continuous, since it is a composition of two continuous functions. Since the coordinate functions of Γ are continuous, Γ is a continuous path joining the points $(\mathbf{u}_1, F(\mathbf{u}_1))$ and $(\mathbf{u}_2, F(\mathbf{u}_2))$ in G, hence G is path connected.

Problem 4. Let $d(n)$ be the number of digits in the decimal expansion of a natural number n, e.g.

$$d(7) = 1,\ d(262) = 3,\ d(20032004) = 8.$$

Determine the interval of convergence (including the behavior at the endpoints) for the series

$$\sum_{n=1}^{\infty} \frac{10^{d(n)}}{n} x^n.$$

Solution. Let $a_n = 10^{d(n)}/n$ be the n-th coefficient of the power series. Suppose

$$10^k \leq n < 10^{k+1}.$$

Then $d(n) = k + 1$ and so

$$1 = \frac{10^{k+1}}{10^{k+1}} < \frac{10^{d(n)}}{n} \leq \frac{10^{k+1}}{10^k} = 10.$$

Accordingly,

$$1 = 1^{1/n} < |a_n|^{1/n} \leq 10^{1/n}$$

Now, $10^{1/n} \to 1$ as $n \to \infty$, so $|a_n|^{1/n} \to 1$ as well. Therefore, the radius of convergence is

$$R = 1/(\lim_{n\to\infty} |a_n|^{1/n}) = 1.$$

The series diverges for $x = \pm 1$ since $a_n \not\to 0$ as $n \to \infty$.

Problem 5. Determine whether the following sequences of functions converge uniformly or pointwise (or neither) in the regions indicated; explain why.

a)

$$f_n(x) = \begin{cases} \frac{\sin nx}{nx}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

for $x \in [-\pi, \pi]$.

b) $f_n(x) = x^2/(3 + 2nx^2)$ for $x \in [0, 1]$.

c) Find $\lim_{n\to\infty} f_n(x)$ in a); is it continuous?

Solution. The limiting function $g = \lim_{n\to\infty} f_n(x)$ in a) is equal to 1 at $x = 0$ and to 0 for $x \neq 0$ (since $|\sin nx| \leq 1$ while $nx \to \infty$ for $0 < |x| \leq \pi$). Since f_n is continuous for every n ($\lim_{x\to 0}(\sin(nx)/(nx)) = 1$), the convergence cannot be uniform, since a uniform limit of continuous functions is continuous. So, the functions in a) converge only pointwise. The functions in b) converge uniformly to the zero function on $[0,1]$. Since $f_n(0) = 0$, there is nothing to prove there. For $0 < x \leq 1$, we can estimate f_n as follows:

$$0 < \frac{x^2}{3 + 2nx^2} = \frac{1}{2n + 3/x^2} < \frac{1}{2n}$$

Accordingly, as $n \to \infty$, $0 \leq f_n(x) \leq 1/(2n)$ and thus converges to zero uniformly by the "squeezing principle".

Problem 6. Determine whether the following sets are open, closed (or neither open nor closed) and explain why.

a) The set of all $(x, y, z) \in \mathbf{R}^3$ such that $|\cos(2x + 3y + 5z)| < 1/2$ and $x^2 + y^2 + z^2 < 180$.

b) The set of all continuous functions $f \in C([0,1])$ (with the uniform distance) such that $|f(1/n)| \leq 1/n^2$ for every natural $n \geq 1$.

Solution. The functions $f_1(x,y,z) = \cos(2x+3y+5z)$ and $f_2(x,y,z) = x^2+y^2+z^2$ are continuous everywhere (by results about the continuity of sum and composition of continuous functions, and since linear and quadratic functions and $\cos x$ are continuous everywhere). Accordingly, the sets $U_1 = f_1^{-1}((-1/2, 1/2))$ and $U_2 = f_2^{-1}((-\infty, 180))$ are open, since they are inverse images of open sets by continuous functions. Accordingly, the set U in a) is open, since it is an intersection of two open sets U_1 and U_2. It is easy to see that U is nonempty and that $U \neq \mathbf{R}^3$. Since $\mathbf{R}^3$ is connected, U cannot be both closed and open, so it is not closed.
For b), let B_k be the set of all continuous functions f on $[0,1]$ such that $|f(1/k)| \leq 1/k^2$ for a *fixed* $k \geq 1$. Then the set V in b) is equal to $\cap_{k=1}^{\infty} B_k$. If we show that B_k is closed for all k, then we can conclude that B is also closed as an intersection of closed sets. The set V is nonempty (the zero function lies in V), and its complement is also nonempty (the function $f(x) \equiv 2000$ is not in V). Since the space of continuous functions on $[0,1]$ is connected (being convex), the set V cannot be both open and closed. It remains to be shown that B_k is closed. Suppose a sequence of functions $f_j \in B_k$ converges to a function f (which is continuous by a theorem about uniform convergence, but we are only considering continuous functions anyway, so we may as well assume that it's continous!). Since $\text{dist}(f_j, f) = \max |f_j(x) - f(x)|$ goes to 0 as $j \to \infty$ by the definition of convergence, we see that $|f_j(1/k) - f(1/k)| \to 0$ as $j \to \infty$. Since the interval $[-1/k^2, 1/k^2]$ is closed, we conclude that $f(1/k) \in [-1/k^2, 1/k^2]$, and so $f \in B_k$ and the set B_k is closed, QED.

Problem 7.

a) Prove that the set A of all $(x, y, z) \in \mathbf{R}^3$ such that $x^2+y^2+z^2 \leq 10$ and $|\exp(5y-3z^2)-1| \leq 1/3$ is compact.

b) Does the function $(x + y^2 + z^3)^2$ attain a maximum and a minimum on the set A? What is the value of the minimum?

Solution. Let A be the set in a). Then A is the intersection of the closed ball U of radius $\sqrt{10}$ centered at 0, with the set $V = g^{-1}([2/3, 4/3])$ where $g(x, y, z) = \exp(5y - 3z^2)$. The function g is continuous (it's a composition of the continuous function exp with a sum of two continuous functions), so the set V is closed, and since U is closed and bounded, the set $A = U \cap V$ is a closed and bounded subset of $\mathbf{R}^3$, and so is compact. The function $f(x, y, z) = (x + y^2 + z^3)^2$ is continuous (it's a composition of the continuous function $w \to w^2$ with a sum of three continuous functions), so it attains a maximum and a minimum on a compact set A. The minimum is equal to 0, since $f(x, y, z) \geq 0$ and f vanishes at $(0, 0, 0) \in A$.

Problem 8.

a) Prove that the mapping $F(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ is continuous in the region $\{(x, y)|x^2 + y^2 \leq 1\}$.

b) Prove that the "northern hemisphere" $N := \{(x, y, z)|x^2 + y^2 + z^2 = 1, z \geq 0\}$ is connected.

c) Prove that the function $f(x, y, z) = e^z \cos x \cos y$ attains the value 2 on the set N.

Solution. The function $1 - x^2 - y^2$ is nonnegative and continuous on the set $D = \{x^2 + y^2 \leq 1\}$ (it's a sum of continuous functions), so its composition with a continuous function $\sqrt{}$ is also continuous on D. The component functions of the mapping F are thus continuous functions, hence F itself is continuous on D. The continuous function F maps the set D into N. The set D is a ball in $\mathbf{R}^2$ and so is convex, path connected and connected. Therefore, its image under a continuous function is also connected. The function f takes the value $e > 2$ at $(0, 0, 1) \in N$, and the value $\cos 1 < 2$ at $(0, 1, 0) \in N$, so by the intermediate value theorem f attains the value 2 on N.

Problem 9. For a metric space X and a positive number r, can one have $\mathcal{B}_r(p) = \mathcal{B}_r(q)$ and yet $p \neq q$?

Solution. This can happen, for example, if the distance d on the metric space X is discrete (i.e. $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$), and the radius of the ball is $r > 1$. Then $B(x, r) = X$ for any $x \in X$.

Problem 10. Is the product of two real-valued uniformly continuous functions again uniformly continuous?

Solution. No, not necessarily. For example, let $f(x) = g(x) = x$. Then f and g are obviously uniformly continuous, but $[f \cdot g](x) = x^2$ is not.

Problem 11.

a) Prove that the mapping $F(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ is continuous in the region $\{(x, y)|x^2 + y^2 \leq 1\}$.

b) Prove that the "northern hemisphere" $N := \{(x, y, z)|x^2 + y^2 + z^2 = 1, z \geq 0\}$ is connected.

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Problem 12. Prove that the *shift map* F of the spaces of sequences is continuous with respect to the l_1 distance d_1:

$$F : (x_1, x_2, x_3, \ldots, x_n, x_{n+1}, \ldots) \to (x_2, x_3, x_4, \ldots, x_{n+1}, x_{n+2}, \ldots),$$

i.e. all coordinates are shifted by 1 to the left. Recall that

$$d_1[(x_1, x_2, \ldots), (y_1, y_2, \ldots)] = \sum_{j=1}^{\infty} |x_j - y_j|.$$

Solution: It suffices to show that $d_1(F(x), F(y)) \le d_1(x, y)$ for infinite sequences $x, y \in l_1$. By definition of the shift map we have

$$d_1(F(x), F(y)) = \sum_{k=2}^{\infty} |x_k - y_k| \le \sum_{k=1}^{\infty} |x_k - y_k| = d_1(x, y).$$

The proves that F is continuous.