McGill University

Math 354: Honors Analysis 3

Differentiation in Banach Spaces: Example

Problem 1 Let $\Phi : C([0,1]) \to C([0,1])$ be given by $\Phi(f) = f^3$. Prove that Φ is differentiable, and compute $D\Phi$.

Solution: Let $f, h \in C([0,1])$. $D\Phi(f)$ (if it exists) applied to h is equal to the *directional derivative* of Φ at f *in the direction* h, and can be found by the same formula as the usual directional derivative in $\mathbf{R}^n$:

$$[D\Phi(f)](h) = \lim_{t\to 0} \frac{1}{t}[\Phi(f+th) - \Phi(f)].$$

Substituting into the definition, we find that $\Phi(f+th)\Phi(f) = (f+th)^3 - f^3 = 3f^2th + 3ft^2h^2 + t^3h^3$. It follows that

$$\frac{1}{t}[\Phi(f+th) - \Phi(f)] = 3f^2h + 3tfh^2 + t^2h^3.$$

Taking the limit $t \to 0$, we find that the directional derivative exists and is equal to

$$[D\Phi(f)](h) = 3f^2h. \tag{1}$$

Accordingly, if $D\Phi(f)$ exists, it is given by (1), so we have "guessed" the differential (but haven't proved that it exists yet!)

Now, we can easily show directly that $D\Phi$ exists and is given by (1). Indeed, from the definition we have

$$\Phi(f+h) - \Phi(f) = 3f^2h + 3fh^2 + h^3.$$

It suffices to show that the map $r(h) = 3fh^2 + h^3$ is little o norm. Accordingly, suppose $||h||_\infty < \epsilon$, and let $||f||_\infty = C$. But it is clear that

$$||3fh^2 + h^3||_\infty \leq \epsilon^2(3C + \epsilon),$$

proving that $r(h)$ is little o norm.