Analysis III
Definitions

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Definition 1. *Let X be a metric space. We define **Distance** $d : X \times X \to \mathbb{R}$ to satisfy*

(i) $\forall x \in X, \quad d(x, x) = 0$

(ii) $\forall x \neq y \in X, \quad d(x, y) > 0$

(iii) $\forall x, y \in X, \quad d(x, y) = d(y, x)$

(iv) $\forall x, y, z \in X, \quad d(x, z) + d(z, y) \geq d(x, y)$

Definition 2. *The **p-adic Norm** is defined as follows*

$$x = p^k \cdot \frac{c}{d}$$

where $k \in \mathbb{Z}$ and

$$||x||_p = p^{-k}$$

Definition 3. *The **p-adic Norm** is defined to be*

$$d_p(x, y) = ||x - y||_p$$

Definition 4. *Suppose that X is a linear space, then we say that the **Norm** of $x \in X$ is a map*

$$|| \cdot || : X \to \mathbb{R}_+$$

such that

(i) $||x|| = 0 \iff x = 0$

(ii) $||t \cdot x|| = |t| \cdot ||x||$

(iii) $||x + y|| \leq ||x|| + ||y||$

Definition 5. *Let $A \subset X$ where X is a metric space. Let $x \in X$ (may or may not be in A). If every ball $B(x,r)$ centred at x of radius r has at least one point from A for any $r > 0$, then this is equivalent to calling x a **Contact Point**. Also, just for notational purposes,*

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

Definition 6. *If $B(x,r)$ for any $r > 0$ has infinitely many points from A, then we say that x is a **Limit Point**.*

Definition 7. *If $\forall\, x \in A, \exists\, r > 0$ s.t.*

$$B(x,r) \cap A = \{x\}$$

Definition 8. *The **Closure** $\overline{A}$ of A is the set of all contact points.*

$$\overline{A} = A \cup \{\textit{Limit points of } A\}$$

Definition 9. *$A, B \subset X$. We say that A is **Dense** in B if*

$$B \subset \overline{A}$$

and it is also true that A is dense if and only if A is dense in X.

Definition 10. *We say that X is **Separable** if X has a countable, dense subset.*

Definition 11. *The **distance from a point** x **to a set** A is equivalent to*

$$d(x,A) = \inf_{a \in A} d(x,a)$$

Definition 12. *Let $A \subset X$. We say that A is **Closed** if $\overline{A} = A$.*

Definition 13. *We say that x is an **Interior Point** of A if and only if*

$$\exists\, r > 0 \ \textit{s.t.} \quad B(x,r) \subset A$$

Definition 14. *We say that A is **Open** if every point in A is an interior point.*

Definition 15. *A collection A_α, for $\alpha \in I$ of open sets is called a **Basis** (of all open sets) if and only if any open set in X is a union of a sub-collection of A_α.*

Definition 16. *X is called **Second Countable** if and only if there is a countable basis of open sets of X.*

Definition 17. *We say that X is **Connected** if and only if any subset $A \subset X$ that is both open and closed is either $\emptyset$ or X.*

Definition 18. *Let d_1, d_2 be two distances on X. We say that d_1 and d_2 are **Equivalent** precisely if there are two constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 < \frac{d_1(x,y)}{d_2(x,y)} < c_2$$

This implies that for $r_2 < r_1 < r_3$, we have

$$B_{d_2}(x, r_2) \subset B_{d_1}(x, r_1) \subset B_{d_2}(x, r_3)$$

Definition 19. *Let X be a set and let $\{A_\alpha\}_{\alpha \in I}$ be a collection of open sets. A **Topological Space** satisfies*

(i) $\emptyset, X$ *are both open.*

(ii)

$$\bigcup_{\alpha \in J} A_\alpha$$

is open.

(iii)

$$\bigcap_{i=1}^{n} A_i$$

is open.

Definition 20. *(More General Definition)*
*X is a **Contact Point** of $A \subset X$ where X is a topological space if every neighbourhood of x contains a point in A.*

Definition 21. *We say that a space X is **Hausdorff** if for any $x, y \in X$ such that $x \neq y$, there exists r_1, r_2 such that*

$$B(x, r_1) \cap B(y, r_2) = \emptyset$$

OR
For any $x \neq y$, there exists open sets U containing x and V containing y such that

$$U \cap V = \emptyset$$

Definition 22. *We say that a topological space X is **Metrizable** if there exists a metric d on X such that open sets defined by d give the same topology on X.*

Definition 23. *Let $f : X \to Y$. We say that f is* ***Continuous*** *at $x \in X$, if*

$$\forall \epsilon > 0, \exists \delta > 0 \; s.t. \; \forall y \in X, d_X(x,y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \epsilon$$

and we say that a f is a ***Continuous Function*** *if it is continuous $\forall x \in X$.*
OR

$$\forall (x_n)_{n=1}^{\infty} \to x \;\; (\lim_{n\to\infty} x_n = x), \lim_{n\to\infty} f(x_n) = f(x)$$

Definition 24. *Let X, Y be metric spaces. The map $f : X \to Y$ is called an* ***Isometry*** *if for all $x_1, x_2 \in X$, we have*

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

Definition 25. *Let X, Y be topological spaces. The map $f : X \to Y$ is a* ***Homeomorphism*** *if*

(i) f is bijective.

(ii) Both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous funcitons.

We say that X, Y are ***Homeomorphic*** *if and only if there exists a homeomorphism $f : X \to Y$. Then open/closed sets, closure, limit points and boundary are all the same for X and Y. Also, continuous functions on X, Y are the same.*

Definition 26. *A sequence (x_n) in a metric space X is a* ***Cauchy Sequence*** *if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n > N$, then*

$$d(x_n, x_m) < \epsilon$$

Definition 27. *We say that a metric space X is* ***Complete*** *if and only if every Cauchy sequence is convergent. That is, if (x_n) is Cauchy, then there exists $z \in X$ such that*

$$d(z, x_n) \to 0$$

as $n \to \infty$.

Definition 28. *We say that two Cauchy sequences (x_n) and (y_n) are* ***Equivalent*** *if and only if*

$$d(x_n, y_n) \to 0$$

as $n \to \infty$.

Definition 29. *If we let Y denote the set of all equivalence classes in Z with respect to $\sim$, then for $(x_n), (y_n)$ Cauchy sequences in X, we define*

$$d\left(\overline{(x_n)}, \overline{(y_n)}\right) = \lim_{n\to\infty} d_X(x_n, y_n)$$

Definition 30. *If X is a metric space and $A : X \to X$, then we say that A is a **Contraction Mapping** if there exists $0 < \alpha < 1$ such that*

$$d(A(x), A(y)) < \alpha \cdot d(x, y)$$

for any $x, y \in X$. It is a fact that if A is a contraction mapping, then A is continuous.

Definition 31. *We say that a map f is **Lipschitz**, denoted $f \in Lip_K$ if*

$$|f(x) - f(y)| < K \cdot |x - y|$$

and if $K < 1$ then f is a contraction mapping.

Definition 32. *Let $A \subset X$ where X is a metric space. We say that A is **Sequentially Compact** if every sequence in A has a subsequence which converges to some $x \in X$ ($x \notin A$ is possible).*

Definition 33. *$A \subset X$ is called an ϵ**-net** if for each $x \in X$, there exists $y \in A$ such that*

$$d(x, y) \leq \epsilon$$

Definition 34. *X is **Totally Bounded** if for each $\epsilon > 0$, there exists a finite ϵ-net in X.*
OR
*X is **Totally Bounded** if for any $\epsilon > 0$, there exists a finite set $x_1, \ldots, x_n$ such that*

$$X \subset B(x_1, \epsilon) \bigcap B(x_2, \epsilon) \bigcap \cdots \bigcap B(x_n, \epsilon)$$

where $n = n(\epsilon)$.

Definition 35. *Let $\mathcal{F} = \{\varphi(x)\}$ be a family (collection) of functions on $[a, b]$. We say that $\mathcal{F}$ is **Uniformly Bounded** if there exists $M > 0$ such that*

$$|\varphi(x)| \leq M, \ \ \forall\, x \in [a, b], \ \ \forall\, \varphi \in \mathcal{F}$$

Definition 36. *We say that the family $\mathcal{F}$ is **Equicontinuous** if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\forall\, x_1, x_2 \in [a, b]$, with $|x_1 - x_2| < \delta$, and $\forall\, \varphi \in \mathcal{F}$, we get*

$$|\varphi(x_1) - \varphi(x_2)| < \epsilon$$

Definition 37. *A subset $A \subseteq X$ is **Sequentially Compact In Itself** if and only if any sequence $(x_n) \in A$ has a convergent subsequence that converges in A.*

Definition 38. *A sequentially compact metric space is called **Compactum**.*