McGill University
Math 354: Honors Calculus 3

Bernstein Approximation Theorem. (S. Drury's Math 354 notes, Theorem 86). Let $f(x) \in C([0,1])$. Let the *n-th Bernstein approximation polynomial* $B_n(f,x)$ be defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

Then $B_n(f,x)$ converges uniformly to $f(x)$ on $[0,1]$ as $n \to \infty$.
Lemma. The following identities hold:

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \\ nx &= \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k}, \\ n(n-1)x^2 &= \sum_{k=0}^{n} k(k-1) \binom{n}{k} x^k (1-x)^{n-k}, \end{aligned} \tag{1}$$

Proof of the Lemma. By the binomial formula, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{2}$$

Substituting $y = 1-x$ into (2) proves the first identity in (1).
Applying the operator $x(d/dx)$ to both sides of (2), we get

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^k y^{n-k}. \tag{3}$$

Substituting $y = 1-x$ into (3) proves the second identity in (1).
Applying the operator $x^2(d^2/dx^2)$ to both sides of (2), we get

$$n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^{n} k(k-1) \binom{n}{k} x^k y^{n-k}. \tag{4}$$

Substituting $y = 1-x$ into (4) proves the third identity in (1).
QED
Proof of the Theorem. We first note that

$$\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \left(x^2 - \frac{2k}{n}x + \frac{k^2-k}{n^2} + \frac{k}{n^2}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Opening the brackets and using (2) we find that the last expression is equal to

$$x^2 - 2x^2 + \frac{n(n-1)}{n^2}x^2 + \frac{n}{n^2}x = \frac{x(1-x)}{n}.$$

Choose $\delta > 0$. Consider all k such that $x - k/n| > \delta$. Then using the previous identity, we find that

$$\sum_{k:|x-\frac{k}{n}|>\delta} \delta^2 \binom{n}{k} x^k (1-x)^{n-k} \leq \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n}. \tag{5}$$

Next, we remark that f is uniformly continuous and bounded, since $[0,1]$ is compact. Using the first identity in (1), we can write

$$f(x) = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k}.$$

It follows that

$$|f(x) - B_n(f,x)| = \left| \sum_{k=0}^{n} \left[f(x) - f\left(\frac{k}{n}\right)\right] \binom{n}{k} x^k (1-x)^{n-k} \right| \leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right)\right| \binom{n}{k} x^k (1-x)^{n-k}. \tag{6}$$

In the last expression, denote by S_1 the sum over all k such that $|x - k/n| > \delta$, and by S_2 the sum over all k such that $|x - k/n| \leq \delta$. It follows from (5) that

$$S_1 \leq \sup_x |f(x) - f(k/n)| \frac{x(1-x)}{n\delta^2} \leq \frac{||f||_\infty}{2n\delta^2},$$

since $0 \leq x(1-x) \leq 1/4$.

Now, given $\epsilon > 0$, choose δ in the definition of the uniform continuity of f such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. It follows that in the second sum S_2 we have $|f(x) - f(k/n)| < \epsilon$. Accordingly,

$$S_2 \leq \epsilon \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \epsilon.$$

Putting together the estimates above, we see that

$$|f(x) - B_n(f,x)| \leq \frac{||f||_\infty}{2n\delta^2} + \epsilon.$$

Choosing n large enough, we can make the right-hand side less than 2ϵ, proving the uniform convergence.

QED