McGill University
Math 354: Honors Analysis 3

Baire's Category Theorem and Uniform Boundedness Principle

I. Baire's Category Theorem.

Theorem 1 (Baire's Category Theorem). (Drury, Theorem 61). Let X be a complete metric space, and let A_k be a closed subset of X with empty interior. Then $X \setminus (\cup_k A_k)$ is dense in X, and in particular $X \neq \cup_k A_k$.

Proof of Theorem 1. Suppose for contradiction that $X \setminus (\cup_k A_k)$ is not dense, so there exists $x_0 \in X$ with $U(x_0, r) \subset \cup_k A_k$ for some $r > 0$; we let $t_0 = r/2$. We construct a sequence x_n as follows: since $U(x_0, t_0) \not\subset A_1$ (otherwise A_1 would have a nonempty interior), there exists $x_1 \in (X \setminus A_1) \cap U(x_0, t_0)$.

Next, let $t_1 = \min\{t_0/2, \text{dist}(x_1, A_1)/4\} > 0$. We next find $x_2 \in (X \setminus A_2) \cap U(x_1, t_1)$ (again such a point exists since A_2 has empty interior). Let $t_2 = \min\{t_1/2, \text{dist}(x_2, A_2)/4\} > 0$. Find $x_3 \in (X \setminus A_3) \cap U(x_2, t_2)$, etc. We let $t_n = \min\{t_{n-1}/2, \text{dist}(x_n, A_n)/4\} > 0$, and we let $x_{n+1} \in (X \setminus A_n) \cap U(x_n, t_n)$.

Since $t_n < t_0/2^n$, it is easy to see that x_n-s form a Cauchy sequence, so $x_n \to x$ for some $x \in X$ as $n \to \infty$. We next show that $x \notin A_k$ for all k. This holds since

$$d(x, x_k) \leq \sum_{n=k}^{\infty} d(x_{n+1}, x_n) \leq \sum_{n=k}^{\infty} t_n \leq \text{dist}(x_k, A_k) \sum_{n=k}^{\infty} 2^{-2-n+k} = \frac{\text{dist}(x_k, A_k)}{2}.$$

Finally, we see that

$$d(x, x_0) \leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \sum_{n=0}^{\infty} t_n \leq r \sum_{n=0}^{\infty} 2^{-1-n} = r,$$

so $x \in U(x_0, r)$ which is assumed to lie in $\cup_k A_k$. Contradiction finishes the proof.

II. Uniform Boundedness Principle.

Let X, Y be Banach (i.e. complete normed linear) spaces (in fact, Y need not be complete). Let $F_n : X \to Y$ be a continuous linear map for each n.

Theorem 2: Uniform Boundedness Principle or Banach-Steinhaus Theorem. Assume that the set $\{F_n(x)\}$ is a bounded subset of Y for every fixed $x \in X$. Then there exists a constant $C < \infty$ such that $||F_n||_{op} \leq C$ for all n.

Proof of Theorem 2. Define A_k to be the set

$$\{x \in X : ||F_n(x)|| \leq k, \forall n \geq 1\}. \tag{1}$$

It is easy to show that

i) A_k is closed: it is equal to $\cap_n F_n^{-1}(\bar{U}(0_Y, k))$ and so is an intersection of closed sets.

ii) A_k is convex: this follows easily from the definition and linearity of F_n-s.

By the assumption of Theorem 2, we have $X = \cup_k A_k$. X is complete, so by Theorem 1 one of the sets A_k has a nonempty interior. This means that there exists $z \in X$ and $r > 0$ such that $U(z, r) \subset A_k$ or, equivalently, for any $x \in U(z, r)$ we have

$$||F_n(x)|| \leq k, \qquad \forall n \in \mathbf{N}.$$

Since $||F_n(x)|| = ||F_n(-x)||$, we see that $U(-z, r) \subset A_k$ as well.

Next, let $x \in X$ with $||x|| \leq r$. Then $x = ((z+x)+(-z+x))/2$. Now, both $z+x$ and $z-x$ belong to A_k, and since A_k is convex, we have $\bar{U}(0_X, r/2) \subset U(0_X, r) \subset A_k$. It follows that $||F_n(x)|| \leq k$ for $||x|| \leq r$, thus $||F_n||_{op} \leq 2k/r$ for all n. This finishes the proof of the Theorem 2.

III. Operator Norm.

Let X and Y be Banach (i.e. complete normed linear) spaces, and let $F : X \to Y$ be a continuous linear map. Recall that an *operator norm* $||F||_{op}$ is defined to be $\sup_{||x|| \leq 1} ||F(x)||$. By Problem 5 in Assignment 3, this defines a norm on the space $\mathcal{CL}(X, Y)$ of continuous linear functionals from X to Y.

Theorem 3. The space $\mathcal{CL}(X, Y)$ is complete with respect to the operator norm.

Proof of Theorem 3. Let F_n be a Cauchy sequence in $\mathcal{CL}(X, Y)$, i.e. for any $\epsilon > 0$ there exists $N > 0$ such that $||F_m - F_n||_{op} \leq \epsilon$ for any $m, n \geq N$. Then for any $x \in X$, we have

$$||F_m(x) - F_n(x)|| \leq \epsilon \cdot ||x|| \tag{2}$$

so $\{F_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, the sequence $F_n(x)$ converges to a limit as $n \to \infty$, which we shall call $F(x)$. It suffices to show that $x \to F(x)$ defines a bounded linear functional and that $||F_n - F||_{op} \to 0$ as $n \to \infty$.

We first show linearity. By linearity of F_n, we get $F_n(t_1 \cdot x + t_2 \cdot y) = t_1 \cdot F_n(x) + t_2 \cdot F_n(y)$. Passing to the limit as $n \to \infty$, we get $F(t_1 \cdot x + t_2 \cdot y) = t_1 \cdot F(x) + t_2 \cdot F(y)$, proving the linearity.

Next, let $||x|| \leq 1$. Then by passing to the limit in (2) we find that $||F_m(x) - F(x)|| \leq \epsilon ||x||$ for $m \geq N$, which shows that $||F_m - F||_{op} \leq \epsilon$. Since ϵ was arbitrary, we find that $||F_n - F||_{op} \to 0$ as $n \to \infty$.

Finally, to prove continuity of F it suffices to show that F is bounded. To do that, let $x \in X$ with $||x|| \leq 1$. Choose N such that $||F_n - F_m||_{op} \leq 1$ for $m, n \geq N$. Since F_N is bounded, we have $||F_N(x)|| \leq C$ for some $C < \infty$. By passing to the limit in (2), we find that

$$||F(x)|| \leq ||F_N(x)|| + ||(F_N - F)(x)|| \leq C + 1,$$

which shows that $||F||_{op} \leq C + 1$ and so F is bounded. This finishes the proof of Theorem 3.