Analysis 3 Notes

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1 Definitions

Metric Space: A **metric space** is an ordered pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}_+$ a distance such that

1. $\mathrm{d}(x, y) \geq 0$, $\mathrm{d}(x, y) = 0 \iff x = y$
2. $\mathrm{d}(x, y) = \mathrm{d}(y, x)$
3. **Triangle Inequality:** $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$

Norm: Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** is a function $\|.\| : X \to \mathbb{R}$ such that

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0_X$
3. $\|\alpha x\| = |\alpha| . \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

Definition: $\ell_p = \{x = (x_1, ..., x_n, \dots) \mid \sum_{k=1}^{n} |x_k|^p < \infty\}$

Definition: $\ell_\infty = \{$ bounded sequences $\} = \{x = (x_1, ..., x_n, \dots) \mid \|x\|_\infty = \sup_k |x_k| < \infty\}$

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Topological Space: A **topological space** is a set X together with a collection $\mathcal{O}$ of subsets of X, called **open sets**, such that :

1. The <u>union</u> of any collection of sets in $\mathcal{O}$ is in $\mathcal{O}$. i.e. If $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{O}$ then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{O}$.
2. The <u>intersection</u> of any <u>finite</u> collection of sets in $\mathcal{O}$ is in $\mathcal{O}$. i.e. If $U_1, \dots, U_n$ are in $\mathcal{O}$, then so is $U_1 \cap \dots \cap U_n$
3. Both $\emptyset$ and X are in $\mathcal{O}$.

Some other terminologies:

- The collection $\mathcal{O}$ of open sets is called a **topology** on X.
- The collection $\mathcal{O}$ of <u>all</u> subsets of X, defines a topology on X called the **discrete topology**.
- The set $\mathcal{O} = \{\emptyset, X\}$ defines a topology, the **trivial topology**.
- If $\mathcal{O}$ and $\mathcal{O}'$ define two topologies on X, with every set that is open in $\mathcal{O}$ topology is also open in the $\mathcal{O}'$ topology, i.e. $\mathcal{O} \subset \mathcal{O}'$, we say that $\mathcal{O}'$ is **finer** than $\mathcal{O}$ an that $\mathcal{O}$ is **coarser** than $\mathcal{O}'$.

Closed Set: A subset A of topological space X is **closed** if its complement $X \backslash A$ is open.

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Let X be a topological space, $A \subset X$ and $x \in X$. Then only one of the following is true for x:

1. $\exists$ an open U in X s.t. $x \in U \subset A$
2. $\exists$ an open V in X s.t. $x \in V \subset X \backslash A$
3. None of the above. $\forall U$ open such that $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap X \backslash A \not\emptyset$

Interior of A := $\{x \in X \mid (1) \text{ holds}\} = A^\circ$

Interior of $X \backslash A$:= $\{x \in X \mid (2) \text{ holds}\}$

Boundary of A := $\{x \in X \mid (3) \text{ holds}\} = \partial A$

Closure of A := $A \cup \partial A$

Limit points of A := $\{x \in X \mid (1) \text{ or } (3) \text{ hold}\}$

Dense: If $\bar{A} = X$ then A is **dense** in X.

Separable Space: A topological space X is separable if it contains a countable dense subset. i.e. $\exists A \subset X$ such that A is countable and $\bar{A} = X$. Or equivalently, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X such that every non-empty open subset of X contains at least one element of the sequence.

Limit: Let $x_1, x_2, x_3, \ldots$ be a sequence of points in X. Then $\lim_{n\to\infty} x_n = y \iff \forall$ open sets U such that $y \in U$, $\exists N > 0$ such that $x_n \in U$ $\forall n > N$.

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Basis for a Topology: A collection $\mathcal{B}$ of open sets in a topological space X is called a **basis** for the topology iff every open set in X is a union of sets in $\mathcal{B}$.

Neighbourhood of x: Any set A such that there exists an open U such that $x \in U$ and $U \subset A$.

Subspace Topology: If $Y \subset X$ then open sets in Y = open sets in $X \cap Y$

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Continuous Functions: Let $f : X \to Y$ where X, Y are topological spaces. Then f is continuous iff $\forall$ open sets U in Y, $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is open in X.

Contact Point: Let $A \subset X$ where X is a metric space. Let $x \in X$ (may or may not be in A). If every ball $B(x, r)$ centred at x of radius r has at least one point from A for any $r > 0$, then this is equivalent to calling x a **Contact Point**. Also, just for notational purposes,
$B(x, r) = \{y \in X \mid \mathrm{d}(x, y) < r\}$
Remark: Any $x \in A$ is a contact point of A.

Dense sets in metric spaces: Suppose $S \subset T$ in the metric space X. Then S is **dense** in T if for any $x \in T$ and all $\varepsilon > 0$, there exists $y \in S$ such that $d(x, y) < \varepsilon$.

Cauchy Sequence: $\{x_n\}_{n=1}^{\infty}$ is cauchy $\iff$ $\forall \varepsilon > 0$,$\exists N \in \mathbb{N}$ such that $\forall m, n > N$ $d(x, y) < \varepsilon$.

Completeness in a metric space : A metric space X is complete $\iff$ every Cauchy sequence in X converges to a limit in X. ($\exists y \in X$ such that $d(x_n, y) \to 0$ as $n \to \infty$)

Cauchy sequences in a Topological Space: In a topological space X, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\iff$ for all open sets $U \subset X$, $\exists N \in \mathbb{N}$ such that $\forall m, n > N$ if $x_n \in U$ then $x_m \in U$.

Pointwise Convergence: A sequence of functions f_n on $[a, b]$ converges pointwise to a function f, if given $x \in [a, b]$, then $\forall \varepsilon > 0$ $\exists N > 0$ such that $\forall n > N$,t $|f_n(x) - f(x)| < \varepsilon$.

Uniform Convergence: Consider $C([a, b], \sup(\ell^{\infty}))$. Convergence for the sup distance is also called **uniform convergence** $\iff$ $f_n \to g$ uniformly in $[a, b]$ $\iff$ $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall n > N$ $\forall x \in [a, b]$, $|f_n(x) - g(x)| < \varepsilon$.($\sup_{x \in [a,b]} |f_n(x) - g(x)| < \varepsilon$).

Contraction Map: Let X be a metric space. $A : X \to X$ is a contraction mapping if $\exists 0 < \alpha < 1$ such that $d(A(x), A(y)) < \alpha d(x, y)$ $\forall x, y \in X$.

Lipschitz $f : [0, 1] \to \mathbb{R}$ is Lipschitz if $\exists K > 0$ such that $|f(y) - f(x)| \leq |y - x|$

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Compactness: A topological space X is called compact if every open cover of X has a finite subcover. That is for any collection of open sets $\{U_\alpha\}_{\alpha \in I}$ such that $X = \bigcup_{\alpha \in I} U_\alpha$, there exists a finite set of indices $\alpha_1, \ldots, \alpha_n \in I$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i}$.
A subset $K \subset X$ is compact if $K \subset \bigcup_{\alpha \in I} U_\alpha$ then $\exists \alpha_1, \ldots, \alpha_n \in I$ such that $K \subset \bigcup_{i=1}^{n} U_{\alpha_i}$

Sequentially Compact Let X be a metric space. Then $A \subset X$ is sequentially compact $\iff$ (x_n) a sequence in A, then $\exists y \in X$ such that $x_{n_k} \to y$ as $n \to \infty$ where (x_{n_k}) is a subsequence of (x_n). That is every infinite sequence has a convergent subsequence.

Product Space: Let X and Y be topological spaces. Then open sets in $X \times Y$ are generated by $U \times V$ where U is open in X and V is open in Y. This is called product topology in Y.

Hausdorff Space: A topological space X is Hausdorff iff $\forall x_1 \neq x_2 \in X$, there exists two disjoint open sets U_1, U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$.

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Let $\mathcal{F} = \{\phi(x)\}$ be a family of continuous functions on a finite interval $[a, b] \subset \mathbb{R}$.

Uniformly Bounded: $\exists M > 0$ such that $\forall \phi \in \mathcal{F}$, $|\phi(x)| \leq M$ $\forall x \in [a, b]$.

Uniformly Equicontinuous: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$, $|\phi(x_1) - \phi(x_2)| < \varepsilon$ is true $\forall \phi \in \mathcal{F}$.

ε-net : $A \subset X$ is called an ε-net if for each $x \in X$, there exists a $y \in A$ such that $d(x, y) < \varepsilon$.
If $M \subset X$, an ε-net for M is a subset $S \subset M$ such that:

$$M \subseteq \bigcup_{x \in S} B(x, \varepsilon)$$

where $B(x, \varepsilon)$ is an open ball centred at x with radius ε.

Totally Bounded: X is totally bounded if for each $\varepsilon > 0$, there exists a finite ε-net in X. In other words, X is totally bounded if for any $\varepsilon > 0$, there exists a finite set $x_1, \ldots, x_n$ such that

$$X \subset B(x_1, \varepsilon) \bigcup \ldots \bigcup B(x_n, \varepsilon)$$

where $B(x, \varepsilon)$ is the closed ball centred at x with radius ε.
A set $S \subset X$ is totally bounded if for each $\varepsilon > 0$ there exists a finite ε-net in S. i.e., S is totally bounded if for every $\varepsilon > 0$, there exists a finite subset $\{s_1, \ldots, s_n\}$ of S such that

$$S \subset B(s_1, \varepsilon) \bigcup \ldots \bigcup B(s_n, \varepsilon)$$

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Connected Sets: A topological space X is connected $\iff$ $X \neq U \cup V$ where U and V are open/closed, non-empty and such that $U \cap V = \emptyset$.

Not Connected Set: X can be written as $X = U \cup V$ where U and V are open, non-empty with $U \cap V = \emptyset$.

Path Connected: A topological space X is path connected, if $\forall x_1, x_2 \in X$, there exists a function $f : [a, b] \to X$ such that $f(a) = x_1$, $f(b) = x_2$ and f is continuous.

Path Component: $P(x) = \{y \in X \mid y \text{ can be connected to } x \text{ by a continuous path}\}$

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Rectangle: A (closed) **rectangle** R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$$

The **volume** of rectangle R is denoted by $|R|$ and given by

$$|R| := (b_1 - a_1)(b_2 - a_2) \dots (b_d - a_d)$$

Cube A cube is a rectangle with sides of equal length. So if $Q \subset \mathbb{R}^d$ is a cube of common side length ℓ, then $|Q| = \ell^d$.

Almost Disjoint: A union of rectangles is said to be **almost disjoint** if the interiors of the rectangles are disjoint.

Exterior Measure: If E is any subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) = \inf_{\bigcup_{j=1}^{\infty} Q_j \supset E} \sum_{j=1}^{\infty} |Q_j|$$

Measurable Set/ Lebesgue Measurable: A subset E of $\mathbb{R}^d$ is **Lebesgue measurable** or simply **measurable**, if for any $\varepsilon > 0$ there exists an open set O with $E \subset O$ and

$$m_*(O - E) \leq \varepsilon$$

If E is measurable, we define its **Lebesgue measure** or **measure** by

$$m(E) := m_*(E)\big(= \inf_{O \supset E} m_*(O)\big)$$

Definition: If $E_1, E_2, \dots$ is a countable collection of subsets of $\mathbb{R}^d$ that increases to E in the sense that $E_1 \subset E_2 \subset \cdots \subset E_k \subset \dots$, and $E = \bigcup_{k=1}^{\infty} E_k$, then we write $E_k \nearrow E$.

Definition: If $E_1, E_2, \dots$ is a countable collection of subsets of $\mathbb{R}^d$ that decreases to E in the sense that $E_1 \supset E_2 \supset \cdots \supset E_k \supset \dots$, and $E = \bigcap_{k=1}^{\infty} E_k$, then we write $E_k \searrow E$.

Symmetric Difference: $E \triangle F = (E - F) \cup (F - E)$

σ-algebra : of sets is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections and, complements. For example, collection of all subsets in $\mathbb{R}^d$ or the set of all measurable sets in $\mathbb{R}^d$, i.e. (σ-algebra of measurable sets).

Borel σ-algebra : Denoted by $\mathcal{B}$ it is the smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called Borel sets. It is the smallest in the sense that if S is any other σ-algebra that contains all open sets in $\mathbb{R}^d$, then necessarily $\mathcal{B} \subset S$. In particular, since open sets are measurable, $\mathcal{B} \subset$ of the σ-algebra of measurable sets.

G_δ**:** countable intersection of open sets, i.e. $G_\delta = \bigcap_{j=1}^{\infty} O_j$.

F_δ**:** countable union of closed sets. $F_\delta = \bigcup_{j=1}^{\infty} F_j$.

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Characteristic Function: The characteristic function of a set E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Step Function: are defined as:

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}$$

where each R_k is a rectangle in $\mathbb{R}^d$ and a_k are constants.

Simple Function:

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}$$

where each E_k is a measurable set of finite measure, and the a_k are constants.

Defintion: A real valued function f is **finite valued** if $-\infty < f(x) < \infty$ for all x.

Measurable Function: A real-valued function f defined on a measurable subset E of $\mathbb{R}^d$ is **measurable** if for all $a \in R$, the set

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\}$$

is measurable. For simplification, we denote $\{f < a\} = \{x \in E \mid f(x) < a\}$.

Almost Everywhere Equality: Two functions f and g defined on a set E are equal **almost everywhere** denoted by

$$f(x) = g(x) \qquad \text{a.e. } x \in E$$

if the set $\{x \in E \mid f(x) \neq g(x)\}$ has measure 0. More generally, a property or statement is said to hold almost everywhere (a.e.) if it is true except on a set of measure 0.

Decomposition of a function into two non-negative functions: Given the real-valued function f, we can decompose it in the following way:

$$f(x) = f^+(x) - f^-(x)$$

where

$$f^+(x) = \max(f(x), 0) \qquad \text{and} \qquad f^-(x) = \max(-f(x), 0)$$

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Canonical from of a simple function: Suppose

$$\phi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$$

The **Canonical form** of ϕ is the unique decomposition where the numbers a_k are distinct and non-zero, and the sets E_k are disjoint. Finding the canonical form of ϕ is straightforward. Since ϕ can only take finitely many distinct non-zero values $c_1, c_2, \ldots, c_M$, we may set $F_k = \{x \mid \phi(x) = c_k\}$, and note that the sets F_k are disjoint. Therefore, $\phi = \sum_{k=1}^{M} c_k \chi_{F_k}$ is the desired canonical form of ϕ.

Lebesgue Integral: If ϕ is a simple function with canonical form $\phi(x) = \sum_{k=1}^{M} c_k \chi_{F_k}(x)$, then we define the **Lebesgue Integral** of ϕ by

$$\int_{\mathbb{R}^d} \phi(x)dx = \sum_{k=1}^{M} c_k m(F_k)$$

If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then $\phi(x)\chi_E(x)$ is also a simple function and we define

$$\int_E \phi(x)dx = \int_{\mathbb{R}^d} \phi(x)\chi_E(x)dx$$

We sometimes represent the Lebesgue integral by:

$$\int_{\mathbb{R}^d} \phi(x)dm(x) = \int \phi(x) = \int \phi$$

Support: The **support** of a measurable function f is defined to be the set of all points where f does not vanish.

$$supp(f) = \{x \mid f(x) \neq 0\}$$

We shall also say that f is **supported** on a set E, if $f(x) = 0$, whenever $x \notin E$.

Lebesgue Integral of a bounded function with $m(supp(f)) < \infty$: Suppose f is bounded function that is supperted on a set of finite measure. We define its **Lebesgue Integral** by

$$\int f = \lim_{n\to\infty} \int \phi_n$$

where $\{\phi_n\}_{n=1}^{\infty}$ is any sequence of simple functions satisfying: $|\phi_n| < M$, each ϕ_n is supported on the support of f, and $\phi_n(x) \to f(x)$ for a.e. x as n tends to infinity.
If E is a subset of $\mathbb{R}^d$ with finite measure, and f is bounded with $m(supp(f)) < \infty$, then it is natural to define:

$$\int_E f(x)dx = \int_{\mathbb{R}^d} f(x)\chi_E(x)dx$$

Lebesgue Integral of Non-Negative functions: Let f be a non-negative measurable function which is not necessarily bounded. f is allowed to be extended valued. We define its **Lebesgue integral** by

$$\int f(x)dx := \sup_g \int g(x)dx$$

where the supremum is taken over all measurable functions g such that $0 \leq g \leq f$, and where g is supported and bounded on a set of finite measure. With the above definition, there are two possible cases; the supremum is either finite, or infinite. In the first case, when $\int f(x)dx < \infty$, we shall say that f is **Lebesgue integrable** of simple **integrable**.
If E is any measurable subset of $\mathbb{R}^d$, and $f \geq 0$, then $f\chi_E$ is also positive, and we define:

$$\int_E f(x)dx = \int f(x)\chi_E(x)dx$$

Definition: We write

$$f_n \nearrow f$$

whenever $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that satisfies

$$f_n(x) \leq f_{n+1}(x) \text{ a.e. } x, \text{ all } n \geq 1 \qquad \text{and} \qquad \lim_{n\to\infty} f_n(x) = f(x) \text{ a.e. } x$$

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Lebesgue Integral of general functions: If f is any real-valued measurable function on $\mathbb{R}^d$, we say that f is **Lebesgue Integrable** if the non-negative measurable function $|f|$ is integrable in the sense of Lebesgue integrability of non-negative functions. We define the **Lebesgue integral** of f by

$$\int f = \int f^+ - \int f^-$$

where

$$f^+(x) = \max(f(x), 0) \qquad \text{and} \qquad f^-(x) = \max(-f(x), 0)$$

It can be shown that the integral of f is independent of the decomposition $f = f_1 - f_2$.

The space $\mathcal{L}^1$ of integrable functions: For any integrable function f on $\mathbb{R}^d$, we define the **norm** of f,

$$\|f\| = \|f\|_{\mathcal{L}^1} = \|f\|_{\mathcal{L}^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)| dx$$

We define $\mathcal{L}^1(\mathbb{R}^d)$ to be the space of all equivalence classes of integrable functions, where we define two functions to be **equivalent** if they agree almost everywhere.

2 Theorems, Lemmas, Propositions

Proposition: If $\mathrm{d}(x,y) = \|x-y\|$ then $\mathrm{d}(.,.)$ is a distance.

Proposition (Vector p-norm): Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $1 \leq p \leq \infty$, then

(a) $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ is a norm.

(b) $\|x\|_\infty = \max_{i=1}^n |x_i|$ is also a norm.

Proposition (Function p-norm): Let f be a continuous function on $[a,b]$. Then $\|f\|\, p := (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}$ is a norm.

Holder and Minkowsky Inequalities

Lemma: If $a, b \geq 0$ and p, q are **conjugate exponents**, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Holder Inequality: Let $x, y \in \mathbb{R}^n$ and $p, q \geq 1$. Then

$$\begin{aligned}\sum_{k=1}^{n} |x_k|.|y_k| &\leq (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}.(\sum_{k=1}^{n} |y_k|^q)^{\frac{1}{q}} \\ &= \|x\|_p \,.\, \|y\|_q\end{aligned}$$

Note: When $p = q = 2$, then Holder inequality $\iff$ Cauchy-Schwartz Inequality

Minkowsky Inequality: Let $x, y \in \mathbb{R}^n$ and $p \geq 1$. Then

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Proposition: Holder and Minkowsky inequalities continue to hold for infinite sequences. In other words, let $x = (x_1, ..., x_n, \dots), y = (y_1, \dots, y_n, \dots)$ and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k|.|y_k| &\leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}.(\sum_{k=1}^{\infty} |y_k|^q)^{\frac{1}{q}} \\ &= \|x\|_p \,.\, \|y\|_q\end{aligned}$$

and

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Proposition: Let f, g be continuous functions on $[a, b]$, then

- **Holder inequality:**

$$\int_a^b |f(x).g(x)|dx \leq \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_a^b |g(x)|^q dx\Big)^{\frac{1}{q}}$$

- **Minkowsky Inequality:**

$$\begin{aligned} \|f+g\|_p = \Big(\int_a^b |f(x)+g(x)|^p dx\Big)^{\frac{1}{p}} &\leq \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}} + \Big(\int_a^b |g(x)|^p dx\Big)^{\frac{1}{p}} \\ &= \|f\|_p + \|g\|_p \end{aligned}$$

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Proposition: Let $x \in \mathbb{R}^n$ and $p \geq 1$. Consider the function $f : p \to \|x\|_p$, i.e. $f(p) = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$. Then f is a monotone decreasing function. Moreover, $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty = \max_{k=1}^n |x_k|$.

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Proposition: Let X be a topological space and A a subset of X. Then

(a) A° is open.

(b) $\bar{A}$ is closed.

(c) A open $\iff$ $A = A^\circ$

(d) B closed $\iff$ $B = \bar{B}$

Proposition: $\mathcal{B}$ is a basis for a topology on $X \iff$ the following two properties hold:

1. Every $x \in X$ lies in some $B \in \mathcal{B}$.
2. $\forall B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \emptyset$ and each point $x \in B_1 \cap B_2$ there exists a set $B_3 \in B$ with $x \in B_3 \subset B_1 \cap B_2$.

Lemma: For a subspace $A \subset X$ which is open in X , a subset $B \subset A$ is open in A if and only if B is open in X . This is also true when 'open' is replaced by closed throughout the statement.

Proposition: Let $f : X \to Y$ where X, Y are topological spaces. Then f is continuous iff $\forall$ closed sets V in Y, $f^{-1}(V) = \{x \in X \mid f(x) \in U\}$ is closed in X.

Proposition: If $f_n(x)$ is continuous and $f_n \to g$ uniformly, then g is also continuous.

Proposition The metric space of continuous functions with respect to ℓ_p distance where $p < \infty$ is not complete.

Theorem: If a metric space (X, d) is not complete, then $\exists$ a metric space Y - **completion of X** - and an **isometry** $\phi : X \to Y$ (i.e. $d_Y(\phi(x_1), \phi(x_2)) = d_X(x_1, x_2)$ where ϕ is a bijection) such that

1. (Y, d_Y) is complete
2. $\phi(X)$ is dense in Y.

Proposition: ℓ_p is complete, $p \geq 1$.

Proposition: $C([a, b], d_\infty)$ is complete.

Completeness Criterion: The metric space X is complete $\iff$ for every sequence of nested closed balls

$$B(x_1, r_1) \supset B(x_2, r_2) \supset \cdots \supset B(x_, r_n) \supset \ldots$$

with $r_n \to 0$, then $\bigcap_{i=1}^{\infty} B(x_i, r_i) \neq \emptyset$

Contraction Mapping Principle: Let X be a complete metric space, and let A be a contraction map. Then $\exists$ a unique $x_0 \in X$ such that $A(x_0) = x_0$. Moreover, $\forall x \in X$, the sequence $A^n(x) \to x_0$ as $n \to \infty$.

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Proposition: A closed subset of a compact space is compact in the subspace topology. That is, in a compact space X, a closed subset $K \subset X$ is compact.

Proposition: Suppose that X is compact space and $f : X \to Y$ is a continuous and onto function. Then $Y = f(X)$ is also compact.

Proposition: If X and Y are compact, the so is their product $X \times Y$ with product topology. By induction, this extends to any finite number of compact sets.(For a subset $X \subset \mathbb{R}^n$ to be bounded means that it lies inside some ball of finite radius centred at the origin).

Proposition: Every compact metric space is separable.

Proposition: Metric spaces are Hausdorff.

Propodition: If a topological space is Hausdorff, then:

1. Points are closed subsets of X.
2. A subspace of a Hausdorff space id also Hausdorff.
3. X_1, X_2 Hausdorff, then $X_1 \times X_2$ is also Hausdorff.

Heine-Borel Theorem: If $X \subset \mathbb{R}^n$, then X is compact iff X is closed and bounded.

Azella-Ascolli: Let $\mathcal{F}$ be a family of functions in $C[a, b]$. Then $\mathcal{F}$ is sequentially compact iff

(1) $\mathcal{F}$ is uniformly bounded.

(2) $\mathcal{F}$ is equicontinuous.

Proposition: Let a_n be a sequence of nonnegative real numbers and let $X = \{x \in \ell^\infty \mid |x_n| \leq a_n, \forall n\}$. Then the following statements are equivalent:

(a) X is a compact subset of ℓ^∞.

(b) $\lim_{n\to\infty} a_n = 0$.

Proposition: Let $X \subset \ell^p$ where $1 \leq p < \infty$. Then X is compact iff the following two conditions hold:

(a) X is a closed and bounded subset of ℓ^p.

(b) For all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall x \in X$ we have $\sum_{n>N} |x_d n|^p < \varepsilon$.

Theorem(Another characterization of compactness): Let X be a complete metric space and $A \subset X$. Then A is sequentially compact iff A is totally bounded.

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Proposition (A property of connected sets): If X is a connected topological space, the only subsets of X that are both open and closed are $\emptyset$ and X.

Theorem: $[a, b] \subset \mathbb{R}$ is connected.

Theorem: Let X be a connected topological space and $f : X \to Y$ a continuous function. Then $f(X)$ is connected.

Theorem: Suppose $f : X \to Y$ is onto and f is continuous. Then

(a) X is connected $\implies$ Y is also connected.

(b) X is path connected $\implies$ Y is also path connected.

Theorem: X is path connected $\implies$ X is connected.

Claim: If $P(x_1) \cap P(x_2) \neq \emptyset$ then $P(x_1) = P(x_2)$.

Lemma: Suppose that $A \subset X$ is connected. Then $\bar{A}$ is also connected.

Lemma: $A \subset X$ is open and closed. Then any connected subset $C \subset X$ such that $C \cap A \neq \emptyset$, must satisfy $C \subset A$.

Intermediate Value Theorem: Suppose X is connected, $f : X \to \mathbb{R}$ is continuous and $a < b$. If $\exists x_1, x_2 \in X$ such that $f(x_1) = a$, $f(x_2) = b$, then $\forall c \in (a, b)$, $\exists y \in X$ such that $f(y) = c$.

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Lemma: If a rectangle is the almost disjoint union of finitely may other rectangles, say $R = \bigcup_{k=1}^{N} R_k$, then

$$|R| = \sum_{k=1}^{N} R_k$$

Lemma: If $R_1, R_2, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|$$

Theorem: Every open set $O \subset \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Theorem: Every open subset $O \subset \mathbb{R}^d$ $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

Examples of Exterior measure:

Example (1) : The exterior measure of a point is 0.

Example (2) : The exterior measure of a closed cube is equal to its volume.

Example (3) : If Q is an open cube, $m_*(Q) = |Q|$

Example (4) : The exterior measure of a rectangle is equal to its volume.

Example (5) : The exterior measure of $\mathbb{R}^d$ is infinite.

Example (6) : The Cantor set has exterior measure 0.

Lemma: For every $\varepsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \varepsilon$$

Properties of Exterior Measure:

Observation (1) : (**Monotonicity**) If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Observation (2) : (**Countable Sub-additivity**) If $E = \bigcup_{j=1}^{\infty} E_j$, then

$$m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$$

or in other words,

$$m_*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m_*(E_j)$$

Observation (3) : If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(O)$, where the infimum is taken over all open sets $O \supset E$.

Observation (4) : If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then

$$m_*(E) = m_*(E_1) + m_*(E_2)$$

Observation (5) : If a set E is the countable union of almost disjoint cubes, $E = \bigcup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|$$

Properties of Measurable Sets:

Property (1) : Every open set in $\mathbb{R}^d$ is measurable.

Property (2) : If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

Property (3) : A countable union of measurable sets is measurable.

Property (4) : Closed sets are measurable. (Suffices to show that compact sets are measurable)

Property (5) : The complement of a measurable set is measurable.

Property (6) : A countable intersection of measurable sets is measurable

Lemma: If F is closed, K is compact and $F \cap K = \emptyset$, then $d(F, K) > 0$.

Theorem: If $E_1, E_2, \ldots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

Corollary: Suppose $E_1, E_2, \ldots$ are measurable subsets of $\mathbb{R}^d$.

(i) If $E_k \nearrow E$, then

$$m(E) = \lim_{N \to \infty} m(E_N)$$

(ii) If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then

$$m(E) = \lim_{N \to \infty} m(E_N)$$

Theorem: Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\varepsilon > 0$:

(i) There exists an open set O with $E \subset O$ and $m(O - E) \leq \varepsilon$.

(ii) There exists a closed set F with $F \subset E$ and $m(E - F) \leq \varepsilon$.

(iii) If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E - K) \leq \varepsilon$.

(iv) If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that

$$m(E \triangle F) \leq \varepsilon$$

Translation Invariance Lebesgue Measure: If E is a measurable set in $\mathbb{R}^d$ and $h \in \mathbb{R}^d$, then the set $E_h = E + h = \{x + h \mid x \in E\}$ is also measurable and $m(E + h) = m(E)$.

Dilation Invariance Lebesgue Measure: Suppose $\delta > 0$ and denote by $\delta E := \{\delta x \mid x \in E\}$. If E is measurable, so is δE and $m(\delta E) = \delta^d m(E)$

Reflexion Invariance of Lebesgue Measure: Whenever E is measurable, so is $-E := \{-x \mid x \in E\}$ and $m(-E) = m(E)$.

Theorem: A subset E of $\mathbb{R}^d$ is measurable

(i) if and only if it differs form a G_δ by a set of measure 0.

(ii) if and only if it differs form an F_δ by a set of measure 0.

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Lemma: f is measurable if and only if $\{x \in E \mid f(x) \leq a\} = \{f \leq a\}$ is measurable for every a. Similarly, f is measurable if and only if $\{f \geq a\}$ (or $\{f > a\}$) is measurable for every a. We can have any combination of strict or weak inequalities we choose.

Properties of Measurable Functions:

Property (1) : The finite-valued function f is measurable if and only if $f^{-1}(O)$ is measurable for every open set O, and if and only if $f^{-1}(F)$ is measurable for every closed set F.

Property (2) : If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measurable.

Property (3) : Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence if measurable functions. Then

$$\sup_n f_n(x) \qquad \inf_n f_n(x) \qquad \lim_{n\to\infty} \sup f_n(x) \qquad \lim_{n\to\infty} \inf f_n(x)$$

are measurable.

Property (4) : If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions, and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists, then f is measurable.

Property (5) : If f and g are measurable, then

(i) The integer powers f^k, $k \geq 1$ are measurable.

(ii) $f + g$ and fg are measurable if both f and g are finite valued.

Property (6) : Suppose f is measureble, and $f(x) = g(x)$ for a.e. x. Then g is measurable.

Theorem: Suppose that f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converges pointwise to f, namely:

$$\phi_k(x) \leq \phi_{k+1}(x), \forall x \qquad \text{and} \qquad \lim_{k\to\infty} \phi_k(x) = f(x), \forall x$$

Theorem: Suppose that f is a measurable function on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ that satisfy:

$$|\phi_k(x)| \leq |\phi_{k+1}(x)|, \forall x \qquad \text{and} \qquad \lim_{k\to\infty} \phi_k(x) = f(x), \forall x$$

In particular, we have $|\phi_k(x)| \leq |f(x)|$ for all x and k.

Theorem: Suppose that f is measurable on $\mathbb{R}^d$. Then there exists a seaquence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges point-wise to $f(x)$ for almost every x.

Egorov: Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ (pointwise) a.e. on E. Given $\varepsilon > 0$, we can find a closed set $A_\varepsilon \subset E$ such that $m(E - A_\varepsilon) \leq \varepsilon$ and $f_k \to f$ uniformly on A_ε. (Every convergent sequence is nearly uniformly continuous.)

Lusin: Suppose f is measurable and finite valued on E with E of finite measure. Then for every $\varepsilon > 0$ there exists a closed set F_ε, with

$$F_\varepsilon \subset E, \qquad \text{and} \qquad m(E - F_\varepsilon) \leq \varepsilon$$

and such that $f|_{F_\varepsilon}$ is continuous. (Every function is nearly continuous.)

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Proposition: The integral of simple functions satisfies the following properties:

(i) Independence of the representation: If $\phi = \sum_{k=1}^{M} a_k \chi_{E_k}$ is any representation of ϕ, then

$$\int \phi = \sum_{k=1}^{M} a_k m(E_k)$$

Notice that $\int \phi$ is defined in terms of the canonical representation of ϕ.

(ii) Linearity: If ϕ and ψ are simple, and $a, b \in \mathbb{R}$, then

$$\int a\phi + b\psi = a \int \phi + b \int \psi$$

(iii) Additivity: If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} \phi = \int_E \phi + \int_F \phi$$

(iv) Monotonicity: If $\phi \leq \psi$ are simple, then

$$\int \phi \leq \int \psi$$

(v) Triangle Inequality: If ϕ is a simple function, then so is $|\phi|$, and

$$\left| \int \phi \right| \leq \int |\phi|$$

Lemma: Let f be a function bounded by M on a set E of finite measure. If $\{\phi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M, supported on E, and with $\phi_n(x) \to f(x)$ for a.e. x, then

(i) The $\lim_{n\to\infty} \int \phi_n$ exists.

(ii) If $f = 0$ a.e., then $\lim_{n\to\infty} \int \phi_n = 0$

Proposition: Suppose f and g are bounded functions supported on sets of finite measure. Then the following properties hold.

(i) If $a, b \in \mathbb{R}$, then

$$\int (af + bg) = a \int f + b \int g$$

(ii) Additivity: If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} f = \int_E f + \int_F f$$

(iii) Monotonicity: If $f \leq g$, then

$$\int f \leq \int g$$

(iv) Triangle Inequality: $|f|$ is also bounded, supported on a set of finite measure, and

$$\left| \int f \right| \leq \int |f|$$

Bounded Convergence Theorem: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that are all bounded by M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$. Then f is measurable, bounded, supported on E for a.e. x and

$$\int |f_n - f| \to 0 \qquad \text{as} \qquad n \to \infty$$

Consequently

$$\int f_n \to \int f \qquad \text{as} \qquad n \to \infty$$

In other words,

$$\lim_{n \to \infty} \int f_n = \int \lim_{n \to \infty} f_n$$

Observation: If $f \geq 0$ is bounded and supported on a set E of finite measure and $f = 0$, then $f = 0$ almost everywhere.

Theorem: Suppose f is Riemann integrable on the closed interval $[a, b]$. Then f is measurable, and

$$\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx$$

Proposition: The Integral of non-negative measurable functions enjoys the following properties:

(i) If $f, g \geq 0$ and $a, b \in \mathbb{R}$, then

$$\int (af + bg) = a \int f + b \int g$$

(ii) Additivity: If E and F are disjoint subsets of $\mathbb{R}^d$ and $f \geq 0$, then

$$\int_{E \cup F} f = \int_E f + \int_F f$$

(iii) Monotonicity: If $0 \leq f \leq g$, then

$$\int f \leq \int g$$

(iv) Triangle Inequality: $|f|$ is also bounded, supported on a set of finite measure, and

$$\Big| \int f \Big| \leq \int |f|$$

(v) If g is integrable and $0 \leq f \leq g$, then f is integrable.

(vi) If f is integrable, then $f(x) < \infty$ for almost every x.

(vii) If $f = 0$, then $f(x) = 0$ for almost every x.

Fatou's Lemma: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions with $f_n \geq 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

or in other words,

$$\int \lim_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Monotone Convergence Theorem: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$. Then

$$\lim_{n\to\infty} \int f_n = \int f \qquad \Big(= \int \lim_{n\to\infty} f_n \Big)$$

Corollary to MCT: Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where $a_k(x) \geq 0$ is measurable for every $k \geq 1$. Then

$$\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx$$

If $\sum_{k=1}^{\infty} \int a_k(x)dx$ is finite, then the series $\sum_{k=1}^{\infty} a_k(x)dx$ converges for a.e. x.

Proposition: The integral of Lebesgue integrable functions is linear, additivem monotonic and satisfies the triangle inequality.

Proposition: Suppose f is integrable on $\mathbb{R}^d$. Then for every $\varepsilon > 0$:

(i) There exists a set of finite measure B (a ball, for example) such that

$$\int_{B^c} |f| < \varepsilon$$

(ii) There is a $\delta > 0$ such that

$$\int_E |f| < \varepsilon \qquad \text{whenever } m(E) < \delta$$

Dominated Convergence Theorem: Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x, as $n \to \infty$. If $|f_n(x)| \leq g(x)$, where g is integrable, then

$$\int |f_n - f| \to 0 \qquad as \qquad n \to \infty$$

and consequently

$$\int f_n \to \int f \qquad as \qquad n \to \infty$$

Proposition: Suppose f and g are two functions in $\mathcal{L}^1(\mathbb{R}^d)$. Then

(i) $\|af\|_{\mathcal{L}^1} = |a| \, \|f\|_{\mathcal{L}^1}$ for all $a \in \mathbb{C}$.

(ii) $\|f+g\|_{\mathcal{L}^1} \leq \|f\|_{\mathcal{L}^1} + \|g\|_{\mathcal{L}^1}$

(iii) $\|f\|_{\mathcal{L}^1} = 0$ if and only if $f = 0$ a.e.

(iv) $d(f,g) = \|f-g\|_{\mathcal{L}^1}$ defines a metric on $\mathcal{L}^1(\mathbb{R}^d)$.

Theorem (Riesz-Fischer): The vector space $\mathcal{L}^1$ is complete in its metric.

Proposition (Invariance Properties of the Lebesgue Integral:): Suppose that f is an integrable function.Then:

(i) $\int_{\mathbb{R}^d} f(x-h) = \int_{\mathbb{R}^d} f$

(ii) $\delta^d \int_{\mathbb{R}^d} f(\delta x)dx = \int_{\mathbb{R}^d} f(x)dx$

(iii) $\int_{\mathbb{R}^d} f(-x)dx = \int_{\mathbb{R}^d} f(x)dx$

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Fubini: We may write

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \qquad \text{where} \qquad d_1 + d_2 = d \qquad \text{and } d_1, d_2 \geq 1$$

A point in $\mathbb{R}^d$ then takes the form (x,y), where $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. If f is a function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the **slice** of f corresponding to $y \in \mathbb{R}^{d_2}$ is the function f^y of $x \in \mathbb{R}^{d_1}$ given by

$$f^y(x) := f(x,y)$$

Similarly, the slice of f for a fixed $x \in \mathbb{R}^{d_1}$ is

$$f_x(y) := f(x,y)$$

Fubini's Theorem: Suppose $f(x,y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:

(i) The slice f^y is integrable on $\mathbb{R}^{d_1}$

(ii) The function $F(y) = \int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$

(iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \Big(\int_{\mathbb{R}^{d_1}} f(x,y)dx \Big) dy = \int_{\mathbb{R}^d} f$$

Clearly, the theorem is symmetric in x and y so we we conclude

$$\int_{\mathbb{R}^{d_1}} \Big(\int_{\mathbb{R}^{d_2}} f(x,y)dy \Big) dx = \int_{\mathbb{R}^d} f$$

In conclusion

$$\int_{\mathbb{R}^{d_2}} \Big(\int_{\mathbb{R}^{d_1}} f(x,y)dx \Big) dy = \int_{\mathbb{R}^{d_1}} \Big(\int_{\mathbb{R}^{d_2}} f(x,y)dy \Big) dx = \int_{\mathbb{R}^d} f$$