§ Regular Problems from Folland

#19 : Categorical Products

Suppose X ~~X~~ and X' are two products of X_α for $\alpha \in I$ meaning that whenever we have morphisms $f_\alpha : Y \longrightarrow X_\alpha$
continuous maps
for some object Y, we can factor through X and
topological space
X' : There are unique morphisms $F : Y \longrightarrow X$ and $F' : Y \longrightarrow X'$ such that $f_\alpha = \pi_\alpha \circ F = \pi'_\alpha \circ F'$

Take $Y = X'$ and $f_\alpha = \pi'_\alpha$ to obtain a unique $F : X' \longrightarrow X$ such that $\pi'_\alpha = \pi_\alpha \circ F$.

Then take $Y = X$ and $f_\alpha = \pi_\alpha$ to obtain a unique $F' : X \longrightarrow X'$ such that $\pi_\alpha = \pi'_\alpha \circ F'$.

$$
\begin{tikzcd}
X \arrow[r, "F'"] \arrow[dr, "\pi_\alpha"'] & X' \arrow[r, "F"] \arrow[d, "\pi'_\alpha"] & X \arrow[dl, "\pi_\alpha"] \\
& X_\alpha &
\end{tikzcd}
$$

Claim : $F \circ F' = id_X$ and $F' \circ F = id_{X'}$, so that F is an isomorphism (homeomorphism) between X' and X.

$\pi_\alpha = \pi'_\alpha \circ F' = (\pi_\alpha \circ F) \circ F' = \pi_\alpha \circ (F \circ F')$

But taking $Y = X$ and $f_\alpha = \pi_\alpha$, we see that there is a unique morphism $\Phi : X \to X$ such that $\pi_\alpha = \pi_\alpha \circ \Phi$, namely $\Phi = id_X$. Therefore $F \circ F' = id_X$.

Similarly, $F' \circ F = id_{X'}$. So F' and F are inverse isomorphisms between X and X'. $\square$

So the product $X = \prod_{\alpha \in I} X_\alpha$ is unique up to a unique isomorphism.

22 Uniform Convergence

Suppose $\sup_{x\in X} \rho(f_n(x), f_m(x)) \xrightarrow[n,m]{} 0$

Fix $x \in X$. $0 \le \rho(f_n(x), f_m(x)) \le \sup_{x\in X} \rho(f_n(x), f_m(x)) \xrightarrow[n,m]{} 0$, so $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Y is complete, so this Cauchy sequence converges to some element which we call $f(x)$. Now unfix $x \in X$, thereby defining a function $f: X \to Y$.

To see that $\sup_{x\in X} \rho(f_n(x), f(x)) \xrightarrow[n]{} 0$, let $\varepsilon > 0$.

By definition of $f(x) = \lim_{m\to\infty} f_m(x)$, we have $\rho(f_m(x), f(x)) < \frac{\varepsilon}{2}$ for sufficiently large m, say all $m \geq N_1(x)$.

Because $\rho(f_n(x), f_m(x)) \le \sup_{x\in X} \rho(f_n(x), f_m(x)) \xrightarrow[n,m]{} 0$, we have

$\sup_{x\in X} \rho(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$ for, say, $m, n \geq N_2$ where N_2 does not depend on x. Combining these with $m = \max(N_1, N_2)$ to get the best of both worlds, we have

$$\begin{aligned} \rho(f_n(x), f(x)) &\le \rho(f_n(x), f_m(x)) + \rho(f_m(x), f(x)) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \quad \text{for } n \geq N_2. \end{aligned}$$

Since x is arbitrary, $\sup_{x\in X} \rho(f_n(x), f(x)) \le \varepsilon$

Since $\varepsilon > 0$ is arbitrary, $\sup_{x\in X} \rho(f_n(x), f(x)) \xrightarrow[n]{} 0$, as required.

- f is unique: If f and g both work, then $0 \le \sup_{x\in X} \rho(f(x), g(x)) \le \sup_{x\in X} \rho(f(x), f_n(x)) + \sup_{x\in X} \rho(f_n(x), g(x)) \xrightarrow[n]{} 0 + 0 = 0$, so $\sup_{x\in X} \rho(f(x), g(x)) = 0$, which implies that ~~each~~ $f = g$.

- If each f_n is continuous, then so is f. Indeed, let $\varepsilon > 0$

$$\begin{aligned} \rho(f(x), f(y)) &\le \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) \\ &\le 2 \sup_{x\in X} \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) \\ &< 2\varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

for y in a sufficiently small neighborhood of x, by continuity of f_n for n so large that $\|f - f_n\|_\infty < \frac{\varepsilon}{3}$

28 The Quotient Topology

(a) $\phi = \pi^{-1}(\phi) \in \mathcal{T}$ because ϕ is an open subset of X.
$\tilde{X} \in \mathcal{T}$ because $\pi^{-1}(\tilde{X}) = X$ is an open subset of X.
Suppose $V_\alpha \in \mathcal{T}$ for $\alpha \in A$, meaning each $\pi^{-1}(V_\alpha)$ is open in X
Then
$$\pi^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \{x \in X \, ; \, \pi(x) \in V_\alpha \text{ for some } \alpha \in A\}$$
$$= \bigcup_{\alpha \in A} \pi^{-1}(V_\alpha)$$
is a union of open subsets of X, which is open, so $\bigcup_{\alpha \in A} V_\alpha \in \mathcal{T}$.
Suppose $V_\alpha \in \mathcal{T}$ for $\alpha = 1, \ldots, n$.
$$\pi^{-1}\left(\bigcap_{\alpha=1}^{n} V_\alpha\right) = \{x \in X \, ; \, \pi(x) \in V_\alpha \text{ for each } \alpha = 1, \ldots, n\}$$
$$= \bigcap_{\alpha=1}^{n} \pi^{-1}(V_\alpha).$$
This is an intersection of finitely many open subsets of X, which is open, so $\bigcap_{\alpha=1}^{n} V_\alpha \in \mathcal{T}$.
Therefore $\mathcal{T}$ is a topology on $\tilde{X}$.

(b) By definition of $\mathcal{T}$, $\pi : X \longrightarrow \tilde{X}$ is continuous.
So if $f : \tilde{X} \longrightarrow Y$ is continuous, then $f \circ \pi$ is a composition of continuous functions, which is continuous.
Now suppose $f \circ \pi$ is continuous. Take an open set $\mathcal{U} \subseteq Y$.
Since π is surjective,
$$f^{-1}(\mathcal{U}) = \{\pi(x) \, ; \, x \in X \text{ and } f(\pi(x)) \in \mathcal{U}\}$$
$$= \pi(\{x \in X \, ; \, f \circ \pi(x) \in \mathcal{U}\})$$
$$= \pi((f \circ \pi)^{-1}(\mathcal{U}))$$
Since $f \circ \pi$ is continuous, $(f \circ \pi)^{-1}(\mathcal{U})$ is an open subset of X, say V.
$f^{-1}(\mathcal{U}) = \pi(V)$ is open in $\tilde{X}$ because $\pi^{-1}\pi(V) = V$ is open.
So f is continuous.

\# 28 (c)

Any space Z is T_1 iff its singletons are closed:

Suppose Z is T_1. Let $z \in Z$. For each $\zeta \neq z$ in Z, the T_1 property gives an open $U_\zeta \subseteq Z$ with $\zeta \in U$ and $z \notin U$.

$Z \setminus \{z\} = \{\zeta \in Z \,;\, \zeta \neq z\} = \bigcup_{\zeta \neq z} U_\zeta$ is a union of open sets, which is open, so $\{z\}$ is closed.

Conversely, suppose $\{z\}$ is a closed subset of Z for each $z \in Z$. Take any two distinct points $w \neq z$. $\{z\}$ is closed, so $Z \setminus \{z\}$ is an open set containing w but not z.

So Z is T_1.

- $A \subseteq \tilde{X}$ is closed iff $\pi^{-1}(A) \subseteq X$ is closed because $\pi^{-1}(\tilde{X} \setminus A) = X \setminus \pi^{-1}(A)$ and the definition of $\mathcal{T}$ makes $\tilde{X} \setminus A$ open iff $\pi^{-1}(\tilde{X} \setminus A)$ is open in X.

- So $\tilde{X}$ is T_1 if and only if every singleton $\{\pi(x)\}$ is closed (π is surjective), which is the case if and only if $\pi^{-1}(\{\pi(x)\})$ is a closed subset of X. $\pi^{-1}(\{\pi(x)\}) = \{y \in X \,;\, \pi(y) = \pi(x)\}$ is none other than the equivalence class of x.

So $\tilde{X}$ is T_1 if and only if every equivalence class is a closed subset of X.

32 Hausdorff $\Longleftrightarrow$ Limits of Nets Are Unique (assuming AC for $\Longleftarrow$)

- Suppose X is Hausdorff. Assume, for a contradiction, that $\{x_\alpha\}_\alpha$ is a net in X converging to two different limits $x \neq y$. By the Hausdorff property, there are disjoint open sets $U, V \subseteq X$ with $x \in U$, $y \in V$.
Since $x_\alpha \to x$, $x_\alpha \in U$ for sufficiently "far out" α, say for $\alpha \geq \alpha_1$.
Since $x_\alpha \to y$, $x_\alpha \in V$ for sufficiently far out α, say for $\alpha \geq \alpha_2$.
Since the indices α form a directed set, there is a head goose γ that is at least as far ~~out~~ as both α_1 and α_2, i.e., $\gamma \geq \alpha_1$ and $\gamma \geq \alpha_2$.
But then $x_\gamma \in U \cap V = \emptyset$, which is a contradiction.

- Suppose X is not Hausdorff. To produce a net with two different limits, let $x \neq y$ be distinct points with no disjoint neighborhoods, i.e., <u>witnesses</u> to the brutal slaughter of the Hausdorff property. Consider the directed set of all pairs (A, B) where $A \subseteq X$ is open and $x \in A$ while $B \subseteq X$ is open and $y \in B$.
The direction is given by $(A,B) \precsim (C,D)$ iff $A \supseteq C$ and $B \supseteq D$.
Since the Hausdorff property <u>fails</u>, $A \cap B \neq \emptyset$ for each such (A,B).
CHOOSE an element $x_{(A,B)}$ of each of these non-empty sets $A \cap B$, thereby defining a net $\{x_{(A,B)}\}$ in X.
$x_{(A,B)} \xrightarrow[(A,B)]{} x$ because for any neighborhood U of x, $x_{(A,B)}$ is in U once $A \subseteq U$, i.e. $(A,B) \succsim (U, X)$, i.e. <u>eventually</u>.
Similarly, $x_{(A,B)} \xrightarrow[(A,B)]{} y$.
So this net converges to two different limits.

#34 Weak Convergence

The weak topology is the smallest one that makes all the functions $f \in \mathcal{F}$ continuous.

If $x_\alpha \xrightarrow[\alpha]{} x$, then $f(x_\alpha) \xrightarrow[\alpha]{} f(x)$ by continuity.

Conversely, suppose $f(x_\alpha) \xrightarrow[\alpha]{} f(x)$ for all $f \in \mathcal{F}$.
Let $\mathcal{U}$ be an open neighborhood of x.
By definition of the weak topology,

$$\mathcal{U} = \bigcup_{j \in J} \bigcap_{k=1}^{n_j} f_{j,k}^{-1}(V_{j,k})$$

for some open sets $V_{j,k}$ in the range of the $f_{j,k}$, i.e. $\mathcal{U}$ is a union of finite intersections of the subsets of $\mathfrak{X}$ that <u>have to be</u> open for each $f \in \mathcal{F}$ to be continuous.

Since $f_{i,k}(x_\alpha) \xrightarrow[\alpha]{} f_{i,k}(x)$, we have

$$f_{i,k}(x_\alpha) \in V_{i,k} \quad \text{for "eventual" } \alpha$$

where $i \in J$ is such that $x \in \bigcap_{k=1}^{n_i} f_{i,k}^{-1}(V_{i,k})$. There is such an $i \in J$ because $x \in \mathcal{U} = \bigcup_{j \in J}$ blah blah blah.

But $f_{i,k}(x_\alpha) \in V_{i,k}$ means $x_\alpha \in f_{i,k}^{-1}(V_{i,k})$.
This holds for all $k = 1, \dots, n_i$, so $x_\alpha \in \bigcap_{k=1}^{n_i} f_{i,k}^{-1}(V_{i,k})$.

Thus $x_\alpha \in \mathcal{U}$ for eventual α.
Since $\mathcal{U}$ was arbitrary, $x_\alpha \xrightarrow[\alpha]{} x$.

- In particular, $\prod_{i \in A} \mathfrak{X}_i$ has the weak topology from $\mathcal{F} = \{\pi_i ; i \in A\}$, so $x_\alpha \xrightarrow[\alpha]{} x$ in the product topology if and only if $\pi_i(x_\alpha) \xrightarrow[\alpha]{} \pi_i(x)$ for all $i \in A$.

#43 $\underline{\{0,1\}^{[0,1[}}$ <u>is compact</u> <u>but not sequentially compact</u>

Let $a_n(x)$ be the n^{th} bit in the binary expansion of $x \in [0,1[$ where we agree that $\frac{1}{2} = 0.1000000\ldots$ rather than $\frac{1}{2} = 0.0111111111111\ldots$, and similarly for the other dyadic rationals $K/2^m$.

$a_n(x)$ is either 0 or 1, so a_n is a function from $[0,1[$ to $\{0,1\}$, i.e. $a_n \in \{0,1\}^{[0,1[}$ for $n=1,2,3,\ldots$

so we have a sequence in $\{0,1\}^{[0,1[}$.

It has no pointwise convergent subsequence.

Indeed, take any subsequence $(a_{n_K})_{K=1}^{\infty}$.

Let $x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$ where $b_n = \begin{cases} 0 & \text{if } n \text{ is not } n_K \text{ for any } K \\ 1 & \text{if } n = n_K \text{ for } K \text{ odd} \\ 0 & \text{if } n = n_K \text{ for } K \text{ even} \end{cases}$

Then $(a_{n_K}(x))_{K=1}^{\infty}$ is the sequence

$$(n_1^{th} \text{ bit of } x,\ n_2^{th} \text{ bit of } x,\ n_3^{th} \text{ bit of } x,\ \ldots)$$
$$= (b_1,\ b_2,\ b_3,\ \ldots)$$
$$= (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots)$$

which does not converge to anything: it just switches between 1 and 0.

So $(a_{n_K})_{K=1}^{\infty}$ does not converge pointwise.

Thus $(a_n)_{n=1}^{\infty}$ has no pointwise convergent subsequence.

$\therefore \{0,1\}^{[0,1[}$ is not sequentially compact.

However, $\{0,1\}$ is compact, so $\{0,1\}^{[0,1[}$ <u>is compact</u> with the product topology = topology of pointwise convergence. So $(a_n)_{n=1}^{\infty}$ <u>does</u> have a pointwise convergent sub<u>net</u>.

$\therefore$ Subnet of a sequence need not be a subsequence.

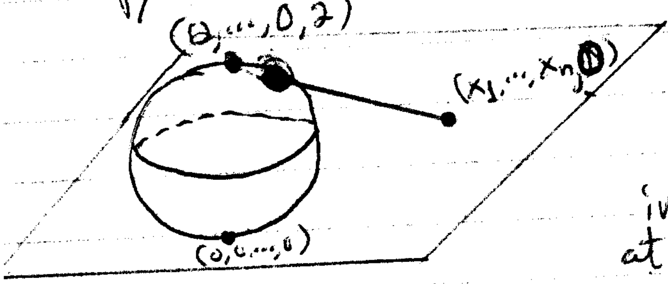
#52 $\underline{\underline{\widehat{\mathbb{R}^n} \cong S^n}}$

$S^n = \{x \in \mathbb{R}^{n+1} ; |x| = 1\} = |\cdot|^{-1}(\{1\})$ is closed because $|\cdot|$ is continuous. It's bounded because it's contained in a (closed) ball of radius 1, or an open ball of radius 2.

By the Heine-Borel Theorem, S^n is compact.

S^n, being a metric space, is Hausdorff.

$\mathbb{R}^n \hookrightarrow S^n$ by stereographic projection and this embedded copy of $\mathbb{R}^n$ covers all of S^n except for one point.



Translate S^n up by one unit. Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, the <u>line</u> from $(x_1, \dots, x_n, 0)$ to $(0, \dots, 0, 2)$ intersects the translated sphere at <u>one other point</u> besides $(0, \dots, 0, 2)$.

Parameterize that line segment:

$$t \cdot (x_1, \dots, x_n, 0) + (1-t) \cdot (0, \dots, 0, 2) \quad , \quad 0 \leq t \leq 1$$

$$= (tx_1, \dots, tx_n, (1-t)2)$$

To be on the sphere: $|(tx_1, \dots, tx_n, 2(1-t)) - (0, \dots, 0, 1)|^2 = 1$

$$t^2(x_1^2 + \dots + x_n^2) + (2(1-t) - 1)^2 = 1$$

$$t^2|x|^2 + 4(1 - 2t + t^2) - 4(1-t) + \cancel{1} = \cancel{1}$$

$$t\,(t(|x|^2 + 4) - 4) = 0$$

$\therefore \underbrace{t = 0}_{\text{north pole}}$ or $t = \dfrac{4}{4 + |x|^2}$

So the embedding is given by translating back down:

$$x \longmapsto \frac{4}{4+|x|^2}(x_1, \dots, x_n, 0) + \frac{|x|^2}{4+|x|^2}(0, \dots, 0, 2) - (0, \dots, 0, 1)$$

- S^n has the three properties which define the one-point compactification uniquely up to homeomorphism, so $S^n \cong \widehat{\mathbb{R}^n}$.

#60 Product of Sequentially Compact Spaces

Suppose X_n is sequentially compact for $n = 1, 2, 3, \ldots$
Take a sequence $\{f_m\}_{m=1}^{\infty}$ in $\prod_{n=1}^{\infty} X_n$, and write
$$f_m = (f_m(1), f_m(2), f_m(3), \ldots) \quad \text{where } f_m(j) \in X_j.$$
$\{f_m(1)\}_{m=1}^{\infty}$ is a sequence in the sequentially compact space X_1, so there is a subsequence $\{f_{m,1}\}$ such that $\{f_{m,1}(1)\}$ converges in X_1.
$\{f_{m,1}(2)\}_{m=1}^{\infty}$ is then a sequence in the sequentially compact space X_2, so passing to a further subsequence $f_{m,2}$ makes $\{f_{m,2}(2)\}_{m=1}^{\infty}$ converge in X_2. $\{f_{m,2}(1)\}_{m=1}^{\infty}$, being a subsequence of $\{f_{m,1}(1)\}_{m=1}^{\infty}$, still converges to the same limit.
Inductively, we obtain an array of subsequences of $\{f_m\}$

$$\begin{matrix} f_{1,1} & f_{2,1} & f_{3,1} & \cdots \\ f_{1,2} & f_{2,2} & f_{3,2} & \\ f_{1,3} & f_{2,3} & f_{3,3} & \\ \vdots & & & \ddots \end{matrix}$$

such that ~~each row~~ $\{f_{m,K}(K)\}_{m=1}^{\infty}$ converges in X_K for each K and each row is a subsequence of the previous row, making $\{f_{m,K}(j)\}_{m=1}^{\infty}$ converge in X_j for $j = 1, \ldots, K-1, K$.
The diagonal sequence $\{f_{m,m}\}_{m=1}^{\infty}$ is then a subsequence of the original $\{f_m\}$ and of EVERY row, so $\{f_{m,m}(K)\}_{m=1}^{\infty}$ converges for each K.
Any given sequence $\{f_m\}$ in $\prod_{n=1}^{\infty} X_n$ has a pointwise convergent subsequence, so $\prod_{n=1}^{\infty} X_n$ is sequentially compact.

#64 Compact Sets of Hölder Continuous Functions

Given a compact metric space (X, ρ), and $\alpha > 0$, let

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha}$$

$$\mathcal{F} = \{ f \in \mathcal{C}(X) \,;\, \|f\|_\infty \le 1 \text{ and } N_\alpha(f) \le 1 \}$$

To see that $\mathcal{F}$ is compact in $\mathcal{C}(X)$, we use Arzelà-Ascoli to prove compactness of $\overline{\mathcal{F}}$ and then check that $\mathcal{F}$ is closed, so that $\mathcal{F} = \overline{\mathcal{F}}$ is compact.

Pointwise Bounded: Let $x \in X$. For any $f \in \mathcal{F}$, $|f(x)| \le \|f\|_\infty \le 1$.

Equicontinuous: Let $\varepsilon > 0$, $x \in X$. For any $f \in \mathcal{F}$, $y \in X$, $y \neq x$,

$$|f(x) - f(y)| = \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \rho(x,y)^\alpha \le N_\alpha(f) \rho(x,y)^\alpha \le \rho(x,y)^\alpha < \varepsilon$$

provided that $\rho(x,y) < \varepsilon^{1/\alpha} = \delta$, so $\mathcal{F}$ is equicontinuous.

By Ascoli, $\overline{\mathcal{F}}$ is compact.

To check that $\mathcal{F}$ is $\|\cdot\|_\infty$-closed, suppose $f_n \in \mathcal{F}$ for $n = 1, 2, 3, \dots$ and $\|f_n - f\|_\infty \xrightarrow[n]{} 0$.

$|f(x)| = |\lim_{n\to\infty} f_n(x)| \le \lim_{n\to\infty} \|f\|_\infty \le 1$ for any $x \in X$, so $\|f\|_\infty \le 1$.

$$\frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} = \frac{|\lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(y)|}{\rho(x,y)^\alpha} = \lim_{n\to\infty} \frac{|f_n(x) - f_n(y)|}{\rho(x,y)^\alpha} \le \lim_{n\to\infty} N_\alpha(f) \le 1$$

for any $x \neq y$, so $N_\alpha(f) \le 1$.

Thus $f \in \mathcal{F}$. So $\mathcal{F}$ is $\|\cdot\|_\infty$-closed.

It follows that $\mathcal{F} = \overline{\mathcal{F}}$ is a compact subset of $\mathcal{C}(X)$.

70 Ideals in $\mathcal{C}(X, \mathbb{R})$, X a compact Hausdorff space

(a) $h(I) = \bigcap_{f \in I} f^{-1}(\{0\})$ is an intersection of closed sets because each $f \in I \trianglelefteq \mathcal{C}(X, \mathbb{R})$ is Continuous, so $h(I)$ is closed.

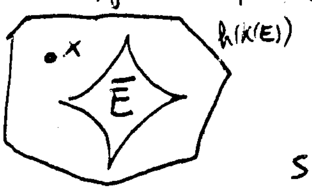
(b) $K(E) = \{ f \in \mathcal{C}(X, \mathbb{R}) ;\ f(x) = 0 \text{ for all } x \in E \}$ is closed because $f_n(x) = 0$ for all $x \in E$, $n = 1, 2, 3, \ldots$ and $\|f_n - f\| \xrightarrow[n]{} 0$ force $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$ for all $x \in E$.

$K(E)$ is an ideal because for any $f_1, f_2 \in K(E)$, $c \in \mathbb{R}$, $g \in \mathcal{C}(X)$, and any $x \in E$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = 0 + 0 = 0$$
$$(cf_1)(x) = c\, f_1(x) = c \cdot 0 = 0$$
$$(f_1 g)(x) = f_1(x) g(x) = 0 \cdot g(x) = 0$$

so $f_1 + f_2$, cf_1 and $f_1 g$ are all in $K(E)$.

(c) $h(K(E))$ is a closed set because it's h of something.
$E \subseteq h(K(E))$ because $f(x) = 0$ for all f such that $f(x) = 0$ for all $x \in E$, when $x \in E$.
For any $x \in \overline{E}$, take a net $\{x_\alpha\}$ in E converging to x.
Let $f \in K(E)$. $f(x_\alpha) = 0$ for each α, so $f(x) = 0$ because continuity of f makes $0 = f(x_\alpha) \xrightarrow[\alpha]{} f(x)$. Since $f \in K(E)$ was arbitrary, $x \in h(K(E))$. So $\overline{E} \subseteq h(K(E))$.
We also could have said $\overline{E} \subseteq h(K(E))$ because $h(K(E))$ is a closed set containing E.
Suppose, for a contradiction, that there is $x \in h(K(E))$ with $x \notin \overline{E}$.



There is a continuous $f : X \to [0,1]$ with $f(x) = 1$ and $f(y) = 0$ on E (and $\overline{E}$), by Urysohn, e.g. But then $f \in K(E)$, so $x \in h(K(E))$ would make $1 = f(x) = 0$ ...

#70 d) If $I \trianglelefteq \mathcal{C}(X, \mathbb{R})$, then $K(h(I)) = \overline{I}$

$h(I)$ is closed, by a), so $\mathcal{U} = X \setminus h(I)$ is open.

Any $f \in K(h(I))$ vanishes on $h(I)$, so it can be thought of as a function on $\mathcal{U}$ that vanishes at ∞:

$K(h(I))$ corresponds to a subalgebra of $\mathcal{C}_0(\mathcal{U}, \mathbb{R})$.

$K(h(I)) = \overline{I}$ follows from Theorem 4.52.

e) $\left\{ \begin{matrix} \text{Closed subsets} \\ \text{of } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Closed ideals} \\ \text{of } \mathcal{C}(X, \mathbb{R}) \end{matrix} \right\}$

$\longrightarrow$ is given by K

$\longleftarrow$ is given by h.

By b), $K(E)$ is a closed ideal for any $E \subseteq X$

By a), $h(I)$ is a closed subset of X for any $I \trianglelefteq \mathcal{C}(X, \mathbb{R})$

When we restrict to <u>closed</u> subsets/ideals, h and K are inverses of each other by (c) and (d):

$$h(K(E)) = \overline{E} = E \quad \text{for closed } E \subseteq X$$

$$K(h(I)) = \overline{I} = I \quad \text{for closed } I \trianglelefteq \mathcal{C}(X, \mathbb{R})$$

So we have a bijection between closed subsets of X and closed ideals of $\mathcal{C}(X, \mathbb{R})$.

§ Extra Problems from Folland

#24 Normal - Urysohn - Tietze. Let X be Hausdorff.

By Urysohn's Lemma, if X is normal (i.e., if the hypothesis of Urysohn's Lemma holds), then the conclusion of Urysohn's Lemma holds.

The proof of Tietze's Extension Theorem in Folland only uses the conclusion of Urysohn's Lemma, so if X satisfies Urysohn then it satisfies Tietze.

$\therefore$ It's enough to prove that X is normal assuming the conclusion of Tietze's Theorem. Then the triad will be complete.

Let A, B be disjoint closed subsets of X.

$A \cup B$, being a finite union of closed sets, is closed.

Let $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$ and well-defined

Since A and B are disjoint, $f : A \cup B \to [0,1]$ is continuous:

$f^{-1}(K) = \begin{cases} A & \text{if } 0 \in K,\ 1 \notin K \\ B & \text{if } 1 \in K,\ 0 \notin K \\ A \cup B & \text{if } 0 \,\&\, 1 \text{ are in } K \\ \emptyset & \text{if neither} \end{cases}$ is a closed subset of $A \cup B$ for any closed $K \subseteq [0,1]$

By the conclusion of Tietze's Theorem, f extends to a continuous $F : X \to [0,1]$.

$U = F^{-1}([0, 1/3[)$ and $V = F^{-1}(]2/3, 1])$ are disjoint open subsets of X with $A \subseteq U$ and $B \subseteq V$.

So X is normal.

#26 Connectedness

a) Suppose X is connected and $f: X \to Y$ is continuous. Take a purported separation $f(X) = A \cup B$ into disjoint non-empty open sets. Then $X = f^{-1}(A) \cup f^{-1}(B)$, so one of $f^{-1}(A)$, $f^{-1}(B)$ is empty. Then either A or B is empty. So $f(X)$ has no <u>real</u> separation and is thus connected.

b) Suppose X is arcwise connected and try to separate it:

$$X = A \cup B.$$

Let $f: [0,1] \to X$ be a path in X. By (a), $f([0,1])$ is connected because the interval $[0,1]$ is connected, so it must lie entirely in A or entirely in B. Whichever it <u>doesn't</u> lie in must be empty or else there would be no path from a point in A to a point in B, contrary to arcwise connectedness. So the separation is phony and X is connected.

Alternatively, fix $a \in X$. For every $x \in X$, there is a path p_x from a to x. Each path p_x is connected, so $X = \bigcup_{x \in X} p_x$ is a union of connected sets with the point a in common. Therefore X is connected.

c) $]0,\infty[$ is connected and $f(x) = \sin(\frac{1}{x})$ is continuous for $x > 0$, so its graph is connected. Similarly, the piece for $-\infty < x < 0$ is connected. Gluing together at $(0,0)$ makes X connected.

However, there is no path from $(0,0)$ to any other point of X since $\sin(\frac{1}{x})$ is not continuous at 0. So X is not arcwise connected.

29 Quotients of $\mathbb{R}^2$ by Matrix Groups

a) $G = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};\ \theta \in \mathbb{R} \right\}$

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ effects a rotation about $(0,0)$ by θ radians.

Taking the quotient of $\mathbb{R}^2$ by G identifies any two points the same distance from the origin, so $\widetilde{\mathbb{R}^2} \cong [0,\infty[$ here because only distance from $(0,0)$ matter.

The mapping $[0,\infty[\longmapsto \widetilde{\mathbb{R}^2}$ sending $r \geq 0$ to the equivalence class of $(r,0)$ is a homeomorphism.

(b) By Problem 28, $\tilde{X}$ is T_1 if and only if all the equivalence classes are closed.

$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + a\begin{bmatrix} y \\ 0 \end{bmatrix}$, so the equivalence class of $\begin{bmatrix} x \\ y \end{bmatrix}$ is a horizontal line through $\begin{bmatrix} x \\ y \end{bmatrix}$ with (horizontal) direction vector $\begin{bmatrix} y \\ 0 \end{bmatrix}$, since all $a \in \mathbb{R}$ are allowed.

Those lines are closed subsets of $\mathbb{R}^2$ (inverse image of $\{0\}$ by same linear functional), so $\widetilde{\mathbb{R}^2}$ is T_1 (except $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ is its own equivalence class, also closed.)

The equivalence class of $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is also a singleton.

$\therefore$ Equivalence classes are lines or points, all closed, so $\widetilde{\mathbb{R}^2}$ is T_1

$\widetilde{\mathbb{R}^2}$ is not Hausdorff. $(0,0)/\sim\ \neq (1,0)/\sim$ but they cannot be separated by disjoint open neighborhoods because $(0,\delta) \sim (1,\delta)$ for any $\delta > 0$, so $(0,\delta)/\sim\ \in \mathcal{U} \cap V$ for any neighborhoods $\mathcal{U}, V$ of $(0,0)/\sim$, $(1,0)/\sim$.

#29 (c) $G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} ; a>0,\ b\in\mathbb{R} \right\}$

$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, so the equivalence class of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the open ray $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} ; a>0 \right\}$, which is not a closed subset of $\mathbb{R}^2$, so $\widetilde{\mathbb{R}^2}$ is not T_1.

For T_0, take $(x,y)/\sim \neq (w,z)/\sim$, so $(x,y) \not\sim (w,z)$. That means there are no $a>0,\ b\in\mathbb{R}$ with

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} w \\ z \end{bmatrix}$$

So $ax+by=w$ and $y=z$ cannot both be true.

If $y\neq z$, then we can separate $(x,y)/\sim$ from $(w,z)/\sim$ by separating ~~2nd~~ 2^{nd} coordinates since the group action only identifies certain points with the same 2^{nd} coordinate.

If $y=z$, then $ax+by\neq w$ for all $a>0,\ b\in\mathbb{R}$.

If T_0 failed, $(x,y)/\sim$ would be in every neighborhood of $(w,z)/\sim\ = (w,y)/\sim$ and vice versa, but that would require $ax+by=w$ for some $a>0,\ b\in\mathbb{R}$, which cannot be. Therefore one can be separated from the other, though we don't know which.

$\therefore\ \widetilde{\mathbb{R}^2}$ is T_0.

d) $(1,1)/\sim\ \neq (0,0)/\sim$ because $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $a,b\neq 0$.

However, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/n \\ 1/n \end{pmatrix}$ for all $n = 1, 2, 3, \dots$,

which means $(1,1)/\sim$ is in every neighborhood of $(0,0)/\sim$ and vice versa, so $\widetilde{\mathbb{R}^2}$ is not T_0.

#38 The Compact Hausdorff Sweet Spot

- By Proposition 4.28, any continuous bijection $f : X \to Y$ from a compact space to a Hausdorff space is a homeomorphism.
- Suppose (X, τ) is a compact Hausdorff space. Assume (X, τ') is still compact with $\tau' \supseteq \tau$.
  $\tau' \supseteq \tau$ makes the identity map
  $$id : (X, \tau') \longrightarrow (X, \tau)$$
  $$id(x) = x$$
  continuous. Id is a bijection, the domain is compact, and the range is Hausdorff, so it's a homeomorphism. It thus induces a bijection between τ and τ'. If the identity map is a bijection between τ and τ' then $\tau = \tau'$.
  So if $\tau' \supsetneq \tau$, then (X, τ') is not compact.
  It is Hausdorff: Hausdorff is about having enough open sets to separate points, and τ' has more open sets than τ.
- Similarly, assume (X, τ') is still Hausdorff with $\tau' \subseteq \tau$. Then the identity map $id : (X, \tau) \longrightarrow (X, \tau')$ is continuous. Therefore it's a homeomorphism. It follows that $\tau' = \tau$.
  So if $\tau' \subsetneq \tau$, then (X, τ') is not Hausdorff.
  It is compact: Any τ'-open cover is a τ-open cover since $\tau' \subseteq \tau$, and thus has a finite subcover by compactness of (X, τ).

#56 An Analysis 2 Problem

a) $\Phi(t) = \dfrac{t}{t+1} = 1 - \dfrac{1}{t+1}$ increasing $\boxed{\checkmark}$

$$\Phi(t+s) = \frac{t+s}{t+s+1} = \frac{t}{t+s+1} + \frac{s}{t+s+1}$$
$$\leq \frac{t}{t+1} + \frac{s}{s+1} \quad \text{for } s,t \geq 0$$
$$= \Phi(t) + \Phi(s)$$

b) • $\Phi(t) = \dfrac{t}{t+1} \leq 1$, so $\Phi \circ \rho$ is bounded.

• If $\Phi(\rho(x,y)) = 0$, then $\rho(x,y) = 0$, so $x = y$.

• $\Phi(\rho(x,y)) = \Phi(\rho(y,x))$ because $\rho(x,y) = \rho(y,x)$ $\boxed{\checkmark}$

• $\Phi(\rho(x,y)) \geq 0$

• $\Phi(\rho(x,z)) \leq \Phi(\rho(x,y) + \rho(y,z))$ by triangle inequality for ρ, since Φ is increasing

$\leq \Phi(\rho(x,y)) + \Phi(\rho(y,z))$ by a) $\boxed{\checkmark}$

So $\Phi \circ \rho$ is a bounded metric on Y.

$\rho(x_n, x) \xrightarrow[n]{} 0$ if and only if $\dfrac{\rho(x_n,x)}{1+\rho(x_n,x)} \xrightarrow[n]{} 0$, so ρ and $\Phi \circ \rho$ give the same convergent sequences.

Thus ρ and $\Phi \circ \rho$ give the same closed sets because $A \subseteq Y$ is closed iff $a_n \to y$ with $a_n \in A$ forces $y \in A$, which is expressed entirely in terms of sequences.

ρ and $\Phi \circ \rho$ give the same closed sets and thus the same topology on Y.

c) Apply b) with $\rho(f,g) = \|f-g\|_\infty$, possibly ∞-valued with $\Phi(\infty) = 1$

$\rho(f_n, f) \xrightarrow[n]{} f$ iff $\|f_n - f\|_\infty \xrightarrow[n]{} 0$, so this is the topology of uniform convergence.

#56 d)

Suppose X is a σ-compact LCH space,

$\mathcal{U}_1 \subseteq \overline{\mathcal{U}}_1 \subseteq \mathcal{U}_2 \subseteq \overline{\mathcal{U}}_2 \subseteq \mathcal{U}_3 \subseteq \overline{\mathcal{U}}_3 \subseteq \ldots$ as in Proposition 4.3

open compact

$X = \bigcup_1^\infty \mathcal{U}_n$

$$\rho(f,g) = \sum_1^\infty 2^{-n} \Phi\left(\sup_{x \in \overline{\mathcal{U}}_n} |f(x) - g(x)| \right)$$

- $\rho(f,g) \geq 0$ ☑
- If $\rho(f,g) = 0$, then $\Phi(\sup_{x \in \overline{\mathcal{U}}_n} |f(x) - g(x)|) = 0$, so $\sup_{x \in \overline{\mathcal{U}}_n} |f(x) - g(x)| = 0$ for all n. Thus $f = g$ on each $\overline{\mathcal{U}}_n$ and hence on $\bigcup_1^\infty \mathcal{U}_n = X$, so $f = g$. ✓
- $\rho(f,g) = \rho(g,f)$ ☑
- Triangle inequality: Each summand satisfies triangle inequality separately, so the sum will as well. ☑

Suppose $f_m \xrightarrow[m]{} f$ uniformly on compact sets. Then $f_m \xrightarrow[m]{} f$ uniformly on each $\overline{\mathcal{U}}_n$, so $\rho(f_m, f) \xrightarrow[m]{} 0$.

Conversely, suppose $\rho(f_m, f) \xrightarrow[m]{} 0$ and take a compact $K \subseteq X$. $X = \bigcup_1^\infty \mathcal{U}_n$, so K is covered by some of the $\mathcal{U}_n$, open! Since K is covered, FINITELY MANY ~~colours~~ $\mathcal{U}_n$ suffice:

$$K \subseteq \bigcup_1^N \mathcal{U}_{n_j}$$

$\rho(f_m, f) \xrightarrow[m]{} 0$ implies uniform convergence on each $\overline{\mathcal{U}}_{n_j}$, hence on each $\mathcal{U}_{n_j}$, and hence on K because the union is finite.

$\therefore$ ρ gives the topology of u.o.c. convergence on $\mathbb{C}^X$.

63 <u>A Compact Integral Operator</u>

We did this in class with Yaiza Canzani.

Tf is continuous:

f is continuous on the compact set $[0,1]$, so it's bounded, say

$$|f(x)| \le M$$

K is continuous on the compact set $[0,1]\times[0,1]$, so it's <u>uniformly</u> continuous. For any given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|K(x,z) - K(y,w)| \le \frac{\varepsilon}{M+1}$$

provided $\sqrt{(x-y)^2+(z-w)^2} < \delta$. In particular,

$$|K(x,z) - K(y,z)| \le \frac{\varepsilon}{M+1} \quad \text{for } |x-y|<\delta, \text{ so}$$

$$\begin{aligned}
|Tf(x) - Tf(y)| &= \left| \int_0^1 K(x,z) f(z)\,dz - \int_0^1 K(y,z) f(z)\,dz \right| \\
&= \left| \int_0^1 \big(K(x,z) - K(y,z)\big) f(z)\,dz \right| \\
&\le \int_0^1 |K(x,z) - K(y,z)| \cdot |f(z)|\,dz \\
&\le \int_0^1 \frac{\varepsilon}{M+1} M\,dz \\
&= \frac{M}{M+1}\varepsilon \\
&< \varepsilon \qquad \text{for } |x-y|<\delta
\end{aligned}$$

So Tf is continuous on $[0,1]$.

Ascoli shows that $\{Tf : \|f\|_\infty \le 1\}$ is precompact in $\mathcal{C}[0,1]$.

Pointwise bounded: $|Tf(x)| \le \int_0^1 |K(x,y)|\cdot|f(y)|\,dy \le \int_0^1 |K(x,y)|\cdot\|f\|_\infty\,dy \le \int_0^1 |K(x,y)|\,dy$ for all such f.

Equicontinuous: Same estimate as above, but with $M = 1$ independent of f, so now the <u>same</u> $\delta > 0$ works for all f with $\|f\|_\infty \le 1$.

By Ascoli, $\{Tf ; \|f\|_\infty \le 1\}$ is precompact.

§ Regular Problems from Dima

Problem 1

$\|f\| \geq 0$ and $=0$ iff $f=0$ ☑

$\|cf\| = |c|\,\|f\|$ because $(cf)^{(K)} = c f^{(K)}$. ☑

Triangle inequality/subadditivity

$$\|f+g\| = \sum_{K=0}^{m} \sup|(f+g)^{(K)}(x)| = \sum_{K=0}^{m} \sup|f^{(K)}(x) + g^{(K)}(x)|$$

$f \mapsto f^{(K)}$ derivatives are linear mappings

$$\leq \sum_{K=0}^{m} \sup\left(|f^{(K)}(x)| + |g^{(K)}(x)|\right)$$

$$\leq \sum_{K=0}^{m} \sup|f^{(K)}(x)| + \sup|g^{(K)}(x)|$$

$$= \|f\| + \|g\|$$ ☑

To see that $(\mathcal{C}^m[0,1], \|\cdot\|)$ is complete and separable, let's cheat. We have an isometric embedding of $\mathcal{C}^m[0,1]$ into the product of $(m+1)$ copies of $\mathcal{C}[0,1]$:

$$\mathcal{C}^m[0,1] \hookrightarrow \mathcal{C}[0,1] \times \dots \times \mathcal{C}[0,1]$$
$$f \longmapsto (f, f', \dots, f^{(m)})$$

where we use the "1-norm" for the product instead of the usual "∞-norm". They're equivalent.

$\mathbb{Q}[x] = \{\text{polynomials over the rationals}\}$ is a countable dense subset of $\mathcal{C}[0,1]$

Finite product of separable spaces is separable because

$$\aleph_0^m = \aleph_0 \quad \text{still countable}$$

(just take product of dense sets).

A subspace of a separable metric space is separable (Analysis 3)

So $\mathcal{C}^m[0,1] \cong$ subspace of separable space $\prod_0^m \mathcal{C}[0,1]$ is separable.

Problem 1 Continued

For completeness, we need to argue that $\mathcal{C}_0^m[0,1]$ embeds as a closed subspace of $\prod_0^m \mathcal{C}[0,1]$.

Suppose $\left(f_n, f_n', \ldots, f_n^{(m)}\right)_{n=1}^{\infty}$ is a sequence in that subspace converging to $(g_0, g_1, \ldots, g_m)$.

This convergence happens componentwise, so $f_n \xrightarrow[n]{} g_0$.

By Analysis 2, $f_n \xrightarrow[n]{} g_0$ uniformly and $f_n' \xrightarrow[n]{} g_1$ uniformly imply $g_0' = g_1$.

Inductively, $g_K = g_0^{(K)}$ for all K up to m.

So the limit is $(g_0, g_0', \ldots, g_0^{(m)})$, which is again in the embedded copy of $\mathcal{C}_0^m[0,1]$, so that copy is closed. A closed subset of a complete space is complete, and a countable product of complete spaces is complete, so $\mathcal{C}_0^m[0,1]$ is complete.

Problem 2

There was a typo on the problem set.
The sum starts at $K=0$ or else $d(f,g)=0 \not\Rightarrow f=g$ since you could add arbitrary constants.

With that modification, the proof that d is a metric is much as in Problem #56: The triangle inequality holds for the sum because it holds for each summand.

Same trick as in Problem 1:

$$\mathscr{C}^\infty[a,b] \hookrightarrow \prod_{K=0}^{\infty} \mathscr{C}[a,b]$$

$$f \longmapsto (f, f', f^{(2)}, f^{(3)}, \ldots)$$

Countable product of complete spaces = complete.

So $\mathscr{C}^\infty[a,b] \cong$ closed subspace of complete space is complete.

Problem 3

Pointwise bounded: $|f_n(x)| \leq M$ $\boxed{\checkmark}$

Equicontinuous: By the Mean Value Theorem,

$$\begin{aligned} |f_n(x) - f_n(y)| &= |f_n'(\xi)(x-y)| \quad \text{for some } \xi \text{ between } x \text{ and } y \\ &\leq M\,|x-y| \\ &< \varepsilon \quad \text{provided } |x-y| < \frac{\varepsilon}{M+1} = \delta, \end{aligned}$$

Same $\delta = \frac{\varepsilon}{M+1}$ works for all f and x, y.

By Ascoli's Theorem, $\{f_n\}_{n=1}^{\infty}$ has a $\|\cdot\|_\infty$-convergent subsequence.

§ Extra Problems from Dima

Problem 4

a) Same as #56.

b) $d_3(x,y) = \min(d_1(x,y), 1) \geq 0$ ☑

$= 0 \iff d_1(x,y) = 0 \iff x = y$ ☑

$\min(d_1(x,y),1) = \min\{d_1(y,x), 1\} = d_3(y,x)$ ☑

Triangle Inequality:

$$d_1(x,z) \leq d_1(x,y) + d_1(y,z)$$

So $d_3(x,z) = \min\{d_1(x,z), 1\} \leq \min\{d_1(x,y) + d_1(y,z), 1\}$

$$\leq \min\{d_1(x,y), 1\} + \min\{d_1(y,z), 1\}$$

$$= d_3(x,y) + d_3(y,z)$$ ☑

$d_3(x_n,x_m) = \min\{d_1(x_n,x_m), 1\} \xrightarrow[n,m]{} \iff d_1(x_n,x_m) \xrightarrow[n,m]{} 0$

So d_3 and d_1 give the same Cauchy sequences.

$d_3(x_n,x) \xrightarrow[n]{} 0 \iff d_1(x_n,x) \xrightarrow[n]{} 0$, so they also give the same convergent sequences and hence same topology.

And of course, $d_3(x,y) = \min\{d_1(x,y), 1\} \leq 1$ is bounded.

c) It's a metric as in Problem 2 and #56

Countable product of complete = complete since each coordinate sequence would be Cauchy.

For a countable dense subset, do the usual Product Topology budget cuts:

$$D = \{(x_1, x_2, x_3, \dots) \,;\, x_j \in \underbrace{D_j}_{\text{countable dense subset of } X_j},\ x_j = 0 \text{ for all but finitely many } j\}$$

Budget cuts $\Rightarrow D = \bigcup_{\text{countable}} \{\text{countable}\}$ finite

$=$ countable union of countable $=$ countable

Problem 5 Metrizing A Product

As in the previous problems, $\rho(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \rho_j(x_j, y_j)$ defines a metric. It's ≤ 1 because $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$

To see that ρ gives the product topology on $\prod_j^{\infty} X_j$, let's use Problem #19 about categorical products.

The mappings $\pi_j : \prod_{k=1}^{\infty} X_k \longrightarrow X_j$

$$\pi_j(x_1, x_2, x_3, \ldots) = x_j$$

are continuous with respect to ρ.

We can factor any continuous maps $Y \xrightarrow{f_i} X_i$ through the metric space $\left(\prod_1^{\infty} X_j, \rho\right)$ to get $f_i = \pi_\alpha \circ F$ for a unique $F : Y \longrightarrow \prod_1^{\infty} X_j$.

By Problem 19, $\left(\prod X_j, \text{product topology}\right) \cong \left(\prod X_j, \text{metric topology}\right)$

We could also have ~~showed~~ shown that the cylinder sets forming a basis for the product topology are all metrically open, and that the balls forming a basis for the metric topology are all product-topologically open.

Problem 6 Sequential Compactness in $\mathcal{C}[0,1]$

a) $\{(ax)^n ; n \in \mathbb{N}, a > 0\}$ is not sequentially compact

The sequence $x, 2x^2, 2^3x^3, 2^4x^4, \dots$ has no $\|\cdot\|_\infty$-convergent subsequence because

$$\|(2x)^n\|_\infty = 2^n \xrightarrow[n]{} \infty .$$

Trouble at $x = 1$.

b) $\{\sin(x+n) ; n \in \mathbb{N}\}$ Yes:

$|\sin(x+n)| \leq 1$

$\left|\frac{d}{dx}\sin(x+n)\right| = |\cos(x+n)| \leq 1$

Apply Problem 3.

c) $\{e^{x-a} ; a > 0\}$ Yes: for $0 \leq x \leq 1$

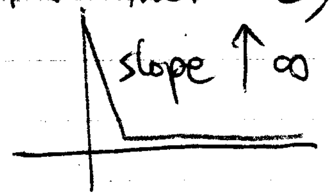
$|e^{x-a}| = e^{-a}e^x \leq e^{-a}e$

$\left|\frac{d}{dx}e^{x-a}\right| = |e^{x-a}| = e^{x-a} \leq e^{-a}e$

Apply Problem 3

d) Yes, by Problem 3. The limit of the subsequence will again be $\mathcal{C}^2$ by $\|\cdot\|_\infty$-convergence, although the strict inequalities $|f'(x)| < B_1$ might become $\leq$.

e) No. $|f(x)| \leq B_0$ and $|f''(x)| \leq B_2$ do not prevent $|f'(x)|$ from being large. For instance, take piecewise cubic approximations to



slope $\uparrow \infty \implies$ not equicontinuous

f) No. $f_n(x) = n$, $n = 1, 2, 3, \dots$

$f_n' = f_n'' = 0$ are bounded

but $1, 2, 3, 4, \dots$ has no convergent subsequence.