McGill University

Math 564, Fall 2011 D. Jakobson

Assignment 5, Part 2 due date to be announced

Problem 1. Let $X = C^m[0,1]$ denote the space of m times continuously differentiable functions on $[0,1]$. Define the norm on X by

$$||f|| = \sum_{k=0}^{m} \max_{x\in[0,1]} |f^{(k)}(x)|.$$

Prove that $(X, ||\cdot||)$ is a complete metric space. Is it separable?

Problem 2. Let $C^\infty[a,b]$ denote the space of infinitely differentiable functions on $[a,b]$ (all the derivatives exist and are continuous). Let

$$d(f,g) = \sum_{k=1}^{\infty} \frac{\max_{x\in[a,b]} |f^{(k)}(x) - g^{(k)}(x)|}{2^k(1 + \max_{x\in[a,b]} |f^{(k)}(x) - g^{(k)}(x)|)}.$$

Prove that d defines the distance on $C^\infty[a,b]$, and that the resulting metric space is complete.

Problem 3. Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of continuously differentiable functions satisfying

$$|f_n(x)| \leq M, |f_n'(x)| \leq M, \qquad \forall x \in [0,1],\ \forall n \in \mathbf{N}.$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence.

Problem 4 (extra credit).

a) Let X be a metric space with the distance d_1. Prove that $d_2(x,y) = d_1(x,y)/(1+d_1(x,y))$ also defines the distance on X. Prove that open sets and Cauchy sequences for d_1 and d_2 coincide.

b) Prove the same results for $d_3(x,y) = \min\{d_1(x,y), 1\}$.

c) Define the distance on the set X of all sequences $x = (x_1, x_2, \ldots)$ of real numbers by the formula

$$d(x,y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}.$$

Prove that d defines a distance on X, and that X is complete with respect to d. Is X separable?

Problem 5 (extra credit). Let $X = \Pi_i X_i$ be a product of metric spaces, with the distance d_i. Consider the product topology on X (a basis of open sets is given by $\Pi_i U_i$, where $U_i = X_i$ except for finitely many i-s). Let $\rho_i = d_i/(1 + d_i)$; it preserves the topology of X_i by Problem 1a. Prove that

$$\rho(x,y) = \sum_{j=1}^{\infty} \frac{\rho_j(x_j, y_j)}{2^j}$$

defines a distance on X, and that the topology given by ρ coincides with the product topology. Hint: Let U be an open set in the basis for the product topology, and let $x \in U$. Prove that there exists $r > 0$ s.t. $B_\rho(x,r) \subset U$. Conversely, let $y \in B_\rho(x,r)$. Prove that there exists a basis set U for the product topology s.t. $y \in U \subset B_\rho(x,r)$.

Problem 6 (extra credit). Determine whether the following sets of functions are sequentially compact in $C[0,1]$:

a) $\{(ax)^n\}, n \in \mathbf{N}, a > 0.$

b) $\{\sin(x+n)\}, n \in \mathbf{N}.$

c) $\{e^{x-a}\}, a > 0.$

d) $\{f \in C^2[0,1] : |f(x)| < B_0, |f'(x)| < B_1, |f''(x)| < B_2\}.$

e) $\{f \in C^2[0,1] : |f(x)| < B_0, |f''(x)| < B_2\}.$

f) $\{f \in C^2[0,1] : |f'(x)| < B_1, |f''(x)| < B_2\}.$