McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 4 — due Friday, October 19

Problem 1 (extra credit). Let $X = C^1[0,1]$ denote the space of continuously differentiable functions on $[0,1]$.

a) Prove that the expression

$$||f||_2 = \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|.$$

defines a norm on X

b) Prove that $(X, ||\cdot||_2)$ is a complete metric space. Is it separable (does it contain a countable dense set)?

c) Prove that $||f||_2$ does not define the same topology on X as the d_∞ norm $\max_{x\in[0,1]} |f(x)|$.

Solution:

a) Clearly $\|\cdot\|_2$ satisfies $\|f\|_2 \geq 0$ and $\|\lambda f\|_2 = |\lambda|\|f\|_2$. If $f \equiv 0$ then $f' \equiv 0$ and $\|f\|_2 = 0$. Now, if $\|f\|_2 = 0$, both $\max_{x\in[0,1]} |f(x)| = 0$ and $\max_{x\in[0,1]} |f'(x)| = 0$. This implies $f \equiv 0$, $f' \equiv 0$ so f is the zero function. The triangle inequality holds since

$$\begin{aligned}\|f+g\|_2 &= \max_{x\in[0,1]} |f+g(x)| + \max_{x\in[0,1]} |(f+g)'(x)| \leq \max_{x\in[0,1]} (|f(x)| + |g(x)|) + \max_{x\in[0,1]} (|f'(x)| + |g'(x)|) \\ &\leq \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)| + \max_{x\in[0,1]} |f'(x)| + \max_{x\in[0,1]} |g'(x)| = \|f\|_2 + \|g\|_2.\end{aligned}$$

b)Let (f_n) be a Cauchy sequence in $\|\cdot\|_2$. Let $\epsilon > 0$. There exists $n \in N$ such that for all $n, m \geq N$

$$|f_n(x) - f_m(x)| + |f_n'(x) - f_m'(x)| < \epsilon$$

for all $x \in [0,1]$. In particular, $d_\infty(f_n', f_m') < \epsilon$, so (f_n') is Cauchy in d_∞, and therefore there exists a uniform limit, say g (continuous). Let

$$f(x) = \int_0^x g(t)dt.$$

We have that, for $n \geq N' \geq N$ and for all $x \in [0,1]$,

$$-\frac{\epsilon}{2} < f_n'(x) - g(x) < \frac{\epsilon}{2}.$$

Integrating over $[0,1]$ we get $-\frac{\epsilon}{2} < f_n(x) - f(x) < \frac{\epsilon}{2}$ and thus

$$\|f_n - f\|_2 < \epsilon.$$

By Stone-Weierstrass theorem we know that polynomials with rational coefficients form a countable dense subset of $C([0,1])$. Let $f \in C^1([0,1])$ and $\epsilon > 0$. We have that $f' \in C([0,1])$ so there exists a polynomial with rational coefficients, $q(x)$, such that

$$d_\infty(f', q) < \frac{\epsilon}{2}.$$

Then, for all $x \in [0,1]$, $-\frac{\epsilon}{2} < f'(x) - q(x) < \frac{\epsilon}{2}$ and therefore $-\frac{\epsilon}{2} < f(x) - Q(x) < \frac{\epsilon}{2}$, where $Q(x)$ is a polynomial with rational coefficients such that $Q'(x) = q(x)$. Then, $\|f - Q\|_2 < \epsilon$.

c)We will show that any ball about $f \equiv 0$ in d_∞ contains functions with arbitrarily big derivative and, therefore, such ball cannot be contained in any ball about 0 in $\|\cdot\|_2$.
Let $\epsilon > 0$ and consider $B^\infty(0,\epsilon)$ the ball about $f \equiv 0$ in d_∞. Let

$$f_n(x) = \frac{1}{n}\sin(n^2 x),$$

$x \in [0,1]$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then for all $n \geq N$, $\|f_n\|_\infty < \epsilon$, so $f_n \in B^\infty(0,\epsilon)$. However,

$$f_n'(x) = n\cos(n^2 x),$$

so $\|f_n\|_2 = n + 1$.

Problem 2. Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of continuously differentiable functions satisfying

$$|f_n(x)| \leq M, |f_n'(x)| \leq M, \qquad \forall x \in [0,1],\ \forall n \in \mathbf{N}.$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence.
Solution. By Arzela-Ascoli theorem, it suffices to show that $\{f_n\}$ is uniformly bounded (true by assumption), and (uniformly) equicontinuous. Accordingly, given $\epsilon > 0$, let $\delta = \epsilon/M$. Then for any n, and for any $x < y \in [0,1]$ such that $|y - x| < \delta$, we have by the intermediate value theorem

$$|f_n(y) - f_n(x)| \leq \delta \cdot \sup_{z \in [x,y]} |f_n'(z)| < M \cdot \frac{\epsilon}{M} = \epsilon,$$

proving uniform equicontinuity. QED
Problem 3. Determine whether the following sets of functions are sequentially compact in $C[0,1]$:

a) $\{(ax)^n\}, n \in \mathbf{N}, a > 0$.

b) $\{\sin(x+n)\}, n \in \mathbf{N}$.

c) $\{e^{x-a}\}, a > 0$.

d) $\{f \in C^2[0,1] : |f(x)| < B_0, |f'(x)| < B_1, |f''(x)| < B_2\}$.

e) (extra credit) $\{f \in C^2[0,1] : |f(x)| < B_0, |f''(x)| < B_2\}$.

f) $\{f \in C^2[0,1] : |f'(x)| < B_1, |f''(x)| < B_2\}$.

Solution:

a) $\{(ax)^n\}, n \in \mathbf{N}, a > 0$. Clearly, for $a > 1$, the sequence $f_n(1) = a^n$ is not bounded, so the answer is NO. Also, for $a < 1$, $f_n(x) \to 0$ uniformly on $[0,1]$, so the answer is YES. If $a = 1$, then $f_n(x) = x^n$ was considered in class. The answer is NO, since the limit function is discontinuous at $x = 1$.

b) $\{\sin(x+n)\}, n \in \mathbf{N}$. The sequence of functions is uniformly bounded, and has uniformly bounded derivatives $f_n'(x) = \cos(x+n)$. The answer is YES by Problem 2.

c) $\{e^{x-a}\}, a > 0$. The sequence of functions is uniformly bounded, and has uniformly bounded derivatives $f_a'(x) = e^{x-a}$. The answer is YES by Problem 2.

d) $\{f \in C^2[0,1] : |f(x)| < B_0, |f'(x)| < B_1, |f''(x)| < B_2\}$. YES by Problem 2.

e) (extra credit) $\{f \in C^2[0,1] : |f(x)| < B_0, |f''(x)| < B_2\}$. The answer is YES. The two conditions imply a uniform bound on the first derivative, then we can use Problem d). The proof will be provided separately.

f) $\{f \in C^2[0,1] : |f'(x)| < B_1, |f''(x)| < B_2\}$. NO, since the sequence is not necessarily uniformly bounded (e.g. arbitrary constant satisfies both conditions, and $f_n(x) = n$ has no convergent subsequence).

Problem 4. Let a_n be a sequence of nonnegative real numbers and let $X = \{x \in l_\infty : |x_n| \leq a_n, \forall n\}$. Prove that the following statements are equivalent:

a) X is a compact subset of l_∞.

b) $\lim_{n\to\infty} a_n = 0$.

Solution. Suppose that $\lim_{n\to\infty} |a_n| \neq 0$ or does not exist. Then there exists $\epsilon > 0$ such that, for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|a_n| \geq \epsilon$. Then we can enumerate the a_{n_k} such that $|a_{n_k}| \geq \epsilon$. For each n_k, define an element $(x_j^{(n_k)})_{j\in\mathbb{N}} \in l^\infty$ by

$$x_j^{(n_k)} = \begin{cases} \epsilon, & j = n_k \\ 0, & j \neq n_k. \end{cases}$$

Then the sequence $x^{(n_k)}$ has no convergent subsequence, and each $x^{(n_k)}$ has $|x_j^{(n_k)}| \leq |a_j|$ for every j, so each $x^{(n_k)} \in X$. Then X is not compact. The contrapositive of what we have just proven implies that if X is compact, then $\lim_{n\to\infty} |a_n| = 0$.

Now suppose that $\lim_{n\to\infty} a_n = 0$. Let $x^{(k)}$ be a sequence in X. Consider $(x_n)_{n\in\mathbb{N}}$ as an element of the product space $\prod_{n\in\mathbb{N}}[-|a_n|, |a_n|]$. Since $[-|a_n|, |a_n|]$ is compact, the product $\prod_{n\in\mathbb{N}}[-|a_n|, |a_n|]$ is compact in the product metric, so that $x^{(k)}$ has a subsequence $x^{(k_j)}$ converging to y. This is equivalent to component-wise convergence, so that for each n,$\lim_{n\to\infty} x_n^{(k_j)} = y_n$. Since each $|x_n^{(k_j)}| \leq a_n$, $|y_n| \leq a_n$ as well, and since this is true for every n, $y \in X$ and thus in l^∞. It remains to show that $\lim_{j\to\infty} \|x^{(k_j)} - y\|_\infty = 0$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\sup_{n\geq N} |a_n| < \epsilon/3$. Since N is finite, there exists J such that, for all $j \geq J$, $\max_{1\leq n\leq N-1} |x_n^{(k_j)} - y_n| < \epsilon/3$, because each $x_n^{(k_j)}$ converges to y_n pointwise. By the triangle inequality,

$$\begin{aligned}
\|x^{(k_j)} - y\|_\infty = \sup_{n\in\mathbb{N}} &\leq \max_{1\leq n\leq N-1} |x_n^{(k_j)} - y_n| + \sup_{n\geq N} |x_n^{(k_j)} - y_n| \\
&< \frac{\epsilon}{3} + \sup_{n\geq N} |x_n^{(k_j)} - y_n| \leq \frac{\epsilon}{3} + \sup_{n\geq N} |x_n^{(k_j)}| + \sup_{n\geq N} |y_n| \\
&\leq \frac{\epsilon}{3} + 2\sup_{n\geq N} |a_n| < \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon
\end{aligned}$$

Then $x^{(k_j)}$ converges to y with respect to $\| \cdot \|_\infty$. Then $x^{(k)}$ has a convergent subsequence, and X is compact.

Problem 5. Let $X \subset l_p$ where $1 \leq p < \infty$. Prove that X is compact if and only if the following two conditions hold:

a) X is a closed and bounded subset of l_p.

b) For all $\epsilon > 0$, there exists $n \in \mathbf{N}$ such that $\forall x \in X$ we have $\sum_{n>N} |x_n|^p < \epsilon$.

Solution.
Suppose that X is compact. As a general property of compact spaces, X is closed and bounded. Given $\epsilon > 0$, choose $x^{(1)}, \ldots, x^{(n)}$ such that $X \subset \cup_{k=1}^{\infty} B(x^{(k)}, \epsilon/2)$. For each k, choose N_k such that $\left(\sum_{n=N_k}^{\infty} |x_n^{(k)}|^p \right)^{1/p} < \epsilon/2$, and let $N = \max\{N_1, \ldots, N_n\}$. Given some arbitrary $x \in X$, $x \in D(x^{(k)}, \epsilon/2)$ for some k. Minkowski's Inequality then implies that

$$\begin{aligned}
\left(\sum_{n=N}^{\infty} |x_n|^p \right)^{1/p} &\leq \left(\sum_{n=N}^{\infty} |x_n - x_n^{(k)}|^p \right)^{1/p} + \left(\sum_{n=N}^{\infty} |x_n^{(k)}|^p \right)^{1/p} < \left(\sum_{n=N}^{\infty} |x_n - x_n^{(k)}|^p \right)^{1/p} + \frac{\epsilon}{2} \\
&\leq \left(\sum_{n=1}^{\infty} |x_n - x_n^{(k)}|^p \right)^{1/p} + \frac{\epsilon}{2} = \|x - x^{(k)}\|_p + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now, suppose that X satisfies properties (a) and (b). Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of points of X. Let $M > 0$ such that, for all $x \in X$, $\|x\|_p \leq M$. Since $\| \cdot \|_p$ is a decreasing function of p, $\| \cdot \|_\infty \leq M$ for every $x \in X$ as well. Consider each $(x_n^{(k)})_{n\in\mathbb{N}}$ as an element of the product space $\prod_{n\in\mathbb{N}}[-M, M]$. Since $[-M, M]$ is compact, and the countable product of compact spaces is compact, $\prod_{n\in\mathbb{N}}[-M, M]$ is also compact. Then $(x^{(k)})$ has a convergent subsequence $(x^{(k_j)})$ converging to y in the product metric on $\prod_{n\in\mathbb{N}}[-M, M]$. This is equivalent to component-wise convergence, so that for each $n \in \mathbb{N}$, $\lim_{j\to\infty} x_n^{(k_j)} = y_n$.

We want to show that $y \in l^p$. For finite N, $\left(\sum_{n=1}^{N} |x_n^{(k_j)}|^p \right)^{1/p} \leq M$. Since the sum is finite, we can take the limit as $j \to \infty$, so that $\left(\sum_{n=1}^{N} |y_n|^p \right)^{1/p} \leq M$. Then $\sum_{n=1}^{N} |y_n|^p$ is an increasing sequence in N and bounded above by M^p, so the limit as $N \to \infty$ exists and

$$\lim_{N\to\infty} \left(\sum_{n=1}^{N} |x_n^{(k_j)}|^p \right)^{1/p} = \|y\|_p \leq M < \infty.$$

Last, we need to show that $\lim_{j\to\infty} \|x^{(k_j)} - y\|_p = 0$. Since X is closed, this implies that $y \in X$, so X is then compact. Let $\epsilon > 0$. First, choose $N_1 \in \mathbb{N}$ such that $\left(\sum_{n=N_1}^{\infty} |y_n|^p \right)^{1/p} < \epsilon/3$. Now choose $N_2 \in \mathbb{N}$ such that, for all $x \in X$, $\left(\sum_{n=N_2}^{\infty} |x_n|^p \right)^{1/p} < \epsilon/3$, and take $N = \max\{N_1, N_2\}$. Finally, choose $J \in \mathbb{N}$ such that, for all $j \geq J$, $\left(\sum_{n=1}^{N-1} |x_n^{(k_j)} - y_n|^p \right)^{1/p} < \epsilon/3$. Such a J exists because the sum is finite and each $x_n^{(k_j)}$ converges to y_n. Then for $j \geq J$, using Minkowski's inequality,

$$\begin{aligned}
\|x^{(k_j)} - y\|_p = \left(\sum_{n=1}^{\infty} |x_n^{(k_j)} - y_n|^p\right)^{1/p} &\leq \left(\sum_{n=1}^{N-1} |x_n^{(k_j)} - y_n|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |x_n^{(k_j)} - y_n|^p\right)^{1/p} \\
&\leq \left(\sum_{n=1}^{N-1} |x_n^{(k_j)} - y_n|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |x_n^{(k_j)}|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |y_n|^p\right)^{1/p} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

So, $x^{(k_j)}$ converges to y in the p-norm, so $x^{(k)}$ has a convergent subsequence.