McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 3 — Solutions to selected problems

Problem 1. Lipschitz functions. Let Lip_K be the set of all functions continuous functions on $[0,1]$ satisfying a *Lipschitz condition with constant* $K > 0$, i.e. such that

$$|f(x) - f(y)| \leq K|x - y| \qquad \forall x, y \in [0,1]. \tag{1}$$

For $f \in Lip_K$, define the norm $||f||$ by

$$||f|| = \sup_{x\in[0,1]} |f(x)| + \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Prove that

i) $||f||$ defines the norm on Lip_K, i.e. $||c \cdot f|| = |c| \cdot ||f||$ and that $||f + g|| \leq ||f|| + ||g||$.

ii) Conclude that $d(f, g) := ||f - g||$ defines a distance on Lip_K.

iii) (**extra credit**) Lip_K is closed, and that it is the closure of the set of all differentiable functions on $[0,1]$ satisfying $|f'(t)| \leq K$.

iv) The set $M = \cup_K Lip_K$ is not closed.

v) (not for credit). What do you think is the closure of M?

Solution: For (i), we remark that linearity is obvious from the definition of $||f||$. The triangle inequality follows from the triangle inequality for the sup-norm, and from taking the supremum over $x \neq y$ in the following inequality:

$$\frac{|f(x) + g(x) - f(y) - g(y)|}{|x - y|} \leq \frac{|f(x) - f(y)| + |g(x) - g(y)|}{|x - y|}.$$

The statement of (ii) follows by a standard argument on how a norm defines a distance.

For (iii), it may be helpful to view Lip_K as a subset of $Lip_{K'}$ for $K < K'$ or as a subset of $C([0,1])$. We want to show that Lip_K is closed in either of those two sets with respect to the usual d_∞ metric. Assume $\{f_n(x)\}$ is a sequence in Lip_K, and that $||f_n(x) - f(x)|| \to 0$ as $n \to \infty$, i.e. that for any $\epsilon > 0$ there exists N such that for all $n \geq N$, we have

$$\sup_x |f_n(x) - f(x)| + \sup_{x\neq y} \frac{|f(x) - f_n(x) - f(y) + f_n(y)|}{|x - y|} < \epsilon. \tag{2}$$

Consider the expression $|f(x) - f(y)|/|x - y|$. It follows from (2) that for $n \geq N$, we have

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \epsilon + \frac{|f_n(x) - f_n(y)|}{|x - y|} \leq \epsilon + K.$$

Since ϵ was arbitrary, we see that $f \in Lip_K$ (and is therefore automatically continuous), hence Lip_K is closed.

Next, if $f \in C^1([0,1])$ with $\sup_t |f'(t)| \le K$, then $f \in Lip_K$ by the intermediate value theorem, since $f(y) - f(x) = (y - x) \cdot f'(\theta)$, where $\theta \in [x, y]$. Next, assume that $f(x)$ satisfies (1) and consider the Bernstein polynomial $B_n(f, x)$. It suffices to show that

Lemma. If $f \in Lip_K$, then $B_n(f, x) \in Lip_K$.

Indeed, B_n is clearly differentiable for all n, approximates f uniformly as $n \to \infty$ by a result proved in class. Also, the inequality $|B_n(f, y) - B_n(f, x)| \le K|y - x|$ implies that $|B_n'(f, x)| \le K$ (if we assume that $|B_n'(f, y)| > K$ for some $y \in [0,1]$, we shall get a contradiction with Lipschitz inequality by choosing x close enough to y and applying the intermediate value theorem).

Proof of the Lemma. (Taken from the note *Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function* by B. Brown, D. Elliott and D. Paget, Journal of Approximation Theory **49**, 196–199, 1987).

Let $0 \le x < y \le 1$. Then

$$
\begin{aligned}
B_n(f, y) &= \sum_{j=0}^{n} \binom{n}{j} (1-y)^{n-j} f\left(\frac{j}{n}\right) (x + (y - x))^j \\
&= \sum_{j=0}^{n} \binom{n}{j} (1-y)^{n-j} f\left(\frac{j}{n}\right) \left[\sum_{k=0}^{j} \binom{j}{k} x^k (y-x)^{j-k}\right] \qquad (3) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{n! x^k (y-x)^{j-k} (1-y)^{n-j}}{k!(j-k)!(n-j)!} f\left(\frac{j}{n}\right)
\end{aligned}
$$

After changing the order of summation and writing $k + l = j$, (3) becomes

$$
B_n(f, y) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n! x^k (y-x)^l (1-y)^{n-k-l}}{k!(l)!(n-k-l)!} f\left(\frac{k+l}{n}\right) \qquad (4)
$$

We next write a similar identity for $B_n(f, x)$:

$$
\begin{aligned}
B_n(f, x) &= \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n}\right) ((y-x) + (1-y))^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n}\right) \left[\sum_{l=0}^{n-k} \binom{n-k}{l} (y-x)^l (1-y)^{n-k-l}\right] \qquad (5) \\
&= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n! x^k (y-x)^l (1-y)^{n-k-l}}{k! l! (n-k-l)!} f\left(\frac{k}{n}\right)
\end{aligned}
$$

Subtracting (5) from (4) we find that

$$
|B_n(f, y) - B_n(f, x)| = \left| \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n! x^k (y-x)^l (1-y)^{n-k-l}}{k! l! (n-k-l)!} \left[f\left(\frac{k+l}{n}\right) - f\left(\frac{k}{n}\right) \right] \right|
$$

By the Lipschitz condition,

$$
\left| f\left(\frac{k+l}{n}\right) - f\left(\frac{k}{n}\right) \right| \le K \left(\frac{l}{n}\right),
$$

so it follows from the preceding inequality that

$$
|B_n(f, y) - B_n(f, x)| \le K \cdot \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n! x^k (y-x)^l (1-y)^{n-k-l}}{k! l! (n-k-l)!} \left(\frac{l}{n}\right).
$$

The last expression is equal to

$$K\sum_{l=0}^{n}\frac{n!(y-x)^l}{l!(n-l)!}\left(\frac{l}{n}\right)\left[\sum_{k=0}^{n-l}\binom{n-l}{k}x^k(1-y)^{n-k-l}\right]=$$
$$K\sum_{l=0}^{n}\binom{n}{l}(y-x)^l\left(\frac{l}{n}\right)(x+1-y)^{n-l}=$$
$$K\cdot B_n(f(z)=z,y-x)=K(y-x),$$

where in the last line we have used the identity $B_n(f(z)=z,x)=x$. That identity follows easily from the second combinatorial identity given in the handout about Bernstein approximation theorem. Summarizing, we have shown that $|B_n(f,y)-B_n(f,x)|\le K(y-x)$, which finishes the proof of the Lemma, as well as the part (iii).

For items (iv) and (v), we note that approximation is understood in terms of the d_∞ (uniform) distance. Thus, it follows from Bernstein approximation theorem that the closure of $\cup_K Lip_K$ is the whole $C([0,1])$. Indeed, for any continuous function f on $[0,1]$, there exists N such that for any $n>N$, $\sup_x|f(x)-B_n(f,x)|\le\epsilon$. Now, we claim that $B_n(f,x)\in\cup_K Lip_K$. Indeed, $|(d/dx)B_n(f,x)|$ is continuous and thus takes a maximum value, say K. As discussed before, that shows that $B_n(f,x)\in Lip_K$. To see that not every continuous function lies in $\cup_K Lip_K$, consider the function $f(x)=\sqrt{x}$. Then $(d/dx)f(x)=1/(2\sqrt{x})$ goes to infinity as $x\to 0$. It is also easy to see that for any $K>0$, there exist $0<x<y<1$ such that

$$\frac{\sqrt{y}-\sqrt{x}}{y-x}=\frac{1}{\sqrt{y}+\sqrt{x}}>K.$$

This happens if x and y are small enough. Thus, $\sqrt{x}\notin Lip_K$ for any K, and hence it is not contained in their union.

Problem 2. Fredholm equation. Use the fixed point theorem to prove the existence and uniqueness of the solution to *homogeneous Fredholm equation*

$$f(x)=\lambda\int_0^1 K(x,y)f(y)dy.$$

Here $K(x,y)$ is a continuous function on $[0,1]^2$ satisfying

$$|K(x,y)|\le M$$

is called the *kernel* of the equation. Consider the mapping of $C([0,1])$ into itself given by

$$(Af)(x)=\lambda\int_0^1 K(x,y)f(y)dy.$$

Let $d=d_\infty$ be the usual “maximum” distance between functions. Prove that

i) Prove that $d(Af,Ag)\le\lambda M\cdot d(f,g)$.

ii) Conclude that A has a unique fixed point in $C([0,1])$ for $|\lambda|<1/M$, e.g. there exists a unique $f\in C([0,1])$ such that $Af=f$.

iii) Prove that f is a solution of the Fredholm equation.

Solution.

For (i), we find that

$$|Af(x) - Ag(x)| = \left|\lambda \int_0^1 K(x,y)(f(xy) - g(y))dy\right| \leq \int_0^1 |\lambda K(x,y)| \cdot |f(y) - g(y)|dy.$$

The last expression is $\leq \int_0^1 |\lambda| M \cdot d(f,g)dy = |\lambda| M d(f,g)$.

For (ii), we remark that it follows from (i) that A is a contraction mapping provided $|\lambda| < 1/M$. The existence of a unique fixed point follows from standard results (we should also mention here that the fixed point theorem depends on the fact that $C([0,1])$ is complete with respect to the d_∞ metric).

Item (iii) follows from the definition of A.

Problem 3. Relative topology. Let X be a metric space, and let Y be a subset of X (with the induced distance). Prove that a set B is open *in* Y if and only if $B = Y \cap A$, where A is open *in* X.

Solution: Suppose $B \subset Y$ is open in Y. So, for every $y \in B$ there exists a positive number r_y such that $U_Y(y, r_y) := \{z \in Y : d(y,z) < r_y\} \subset B$. Let

$$A = \cup_{y \in B} U_X(y, r_y).$$

Clearly, A is an open subset of X (it is a union of open balls). Also,

$$A \cap Y = \cup_{y \in B}(U_X(y, r_y) \cap Y) = \cup_{y \in B} U_Y(y, r_y) = B,$$

since $U_Y(y, r_y) \subset B$.

For the converse, let V be open in X, and let $y \in V \cap Y$. Then $U_X(y,t) \subset V$ for some $t > 0$. But $U_Y(y,t) = U_X(y,t) \cap Y \subset V \cap Y$. Thus, $V \cap Y$ is open in Y, QED.

Problem 4. Let X be a topological space, $X = \cup_n X_n$, where X_n is open for all n. Suppose that the restriction $f|X_n$ is continuous for all n; prove that f is continuous on X.

Solution. Let $f : X \to Y$ be our map, and let $V \subset Y$ be open. To prove continuity, we have to show that $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is open. Now,

$$f^{-1}(V) = f^{-1}(V) \cap (\cup_n X_n) = \cup_n f^{-1}(V) \cap X_n.$$

But the latter set is the preimage of V under the restriction $f|X_n$, and hence is open since $f|X_n$ is continuous for all n. Thus, $f^{-1}(V)$ is open as a union of open sets.

A different proof uses sequences in the case where X is a metric space. Let $x_n \to y \in X$. We want to show that $f(x_n) \to f(y)$. Since open sets X_m cover the whole space, we have $y \in X_m$ for some m. Since X_m is open, we have $B(y,r) \subset X_m$ for some $r > 0$. It follows that $x_n \in B(y,r)$ for large enough n, so $x_n \in X_m$. Since $f|_{X_m}$ is continuous, we have $f(x_n) \to f(y)$, QED.

Problem 5. Consider $C([a,b])$, the vector space of all continuous functions on $[a,b]$, equipped with the usual norm $||f||_p, 1 \leq p \leq \infty$. Consider a map $\Phi : C([a,b]) \to C([a,b])$ defined by $\Phi(f) = f^2$. For what values of p is this map continuous? Please justify carefully your answer.

Solution: We have $\Phi(f+h)-\Phi(f)=2fh+h^2$. First let $p=\infty$. Let $f\in C([a,b])$, denote $||f||_\infty=M$, and choose $0<\epsilon<1/3$. Let $||h||_\infty<\min(\epsilon/(3M),\epsilon)$. Then

$$||2fh+h^2||_\infty \le \frac{2M\epsilon}{3M}+\epsilon^2=\epsilon(2/3+\epsilon)<\epsilon,$$

hence Φ is continuous at f, hence Φ is continuous for $p=\infty$.

Consider next $1\le p<\infty$, and assume WLOG that $[a,b]=[0,1]$. Let $f(x)\equiv 0$, then $\Phi(f+h)-\Phi(f)=h^2$. We would like to choose $h\in C([a,b])$ such that $||h||_p$ is small, but $||h^2||_p$ is large to prove that Φ is not continuous at f.

Let $h(x)=\delta n^{1/p}$ for $x\in[0,1/n]$, $h(x)=0$ for $x\in[2/n,1]$, and let $h(x)$ be linear for $x\in[1/n,2/n]$. Then it is easy to show that

$$\delta^p\le\int_0^1(h(x))^p dx\le 2\delta^p,$$

hence $\delta\le||h||_p\le\delta\cdot 2^{1/p}$, and the expression goes to 0 as $\delta\to 0$.

On the other hand, it is also easy to show that

$$\delta^{2p}n\le\int_0^1(h(x))^{2p}dx\le 2\delta^{2p}n,$$

hence $\delta^2\cdot n^{1/p}\le||h^2||_p\le\delta^2\cdot(2n)^{1/p}$, and the expression diverges as $n\to\infty$, showing that Φ is not continuous at $f\equiv 0$.

Problem 6. Let M be a bounded subset in $C([0,1])$. Prove that the set of functions

$$F(x)=\int_0^x f(t)dt, \qquad f\in M \tag{6}$$

has compact closure (in the space of continuous functions with the uniform distance d_∞).

Solution: Let $\mathcal{F}$ be a family of functions defined by (6). By Arzela-Ascoli Theorem, it suffices to show that $\mathcal{F}$ is bounded and equicontinuous. We first remark, that since M is bounded in $C([0,1])$ (with d_∞, or uniform, distance), there exists $C>0$ such that $|f(t)|<C$ for all $t\in[0,1]$ and for all $f\in M$. It follows that for any $x\in[0,1]$ and for any $f\in M$,

$$|F(x)|=\left|\int_0^x f(t)dt\right|\le\int_0^x|f(t)|dt\le Cx\le C,$$

so $\mathcal{F}$ is bounded in $C([0,1])$. To prove that $\mathcal{F}$ is equicontinuous, we fix $\epsilon>0$, and let $\delta=\epsilon/C$. Suppose $|x-y|<\delta$. Assume (without loss of generality) that $x<y$. Then for any $F\in\mathcal{F}$,

$$|F(y)-F(x)|=\left|\int_x^y f(t)dt\right|\le\int_x^y|f(t)|dt<\delta\cdot C=\epsilon,$$

so $\mathcal{F}$ is equicontinuous. This finishes the proof.

Problem 7. Let X be a compact metric space with a countable base, and let $A:X\to X$ be a map satisfying $d(Ax,Ay)<d(x,y)$ for all $x,y\in X$. Prove that A has a unique fixed point in X.

Solution. Consider the function $f(x) = d(x, Ax)$. We first show that f is continuous. Indeed, if $d(y, x) < \epsilon$, then $d(Ay, Ax) < \epsilon$ as well since A is contracting, therefore as a result of applying the triangle inequality we get $|d(x, Ax) - d(y, Ay)| < 2\epsilon$. Since ϵ was arbitrary, continuity follows. A continuous function on a compact set attains its minimum, say at a point $y \in X$. Suppose that the minimum is positive, i.e. that $Ay \neq y$. Then $d(A^2y, Ay) < d(Ay, y)$ since A is contracting, which contradicts the minimality. Therefore, $d(a, Ay) = 0$ and so y is a fixed point. Uniqueness follows from the contracting property of A in the usual way.

Problem 8 (extra credit). Give an example of a *non-compact* but *complete* metric space X and a map $A : X \to X$ as in Problem 7 such that A doesn't have a fixed point.
Solution. Consider $X = \mathbf{N}$ with $d(m, n) = 1 + 1/(m + n)$. That distance defines discrete topology in X, so X is certainly complete (any Cauchy sequence is eventually constant). Consider any increasing map of $\mathbf{N} \to \mathbf{N}$, for example $A(n) = n^2 + 1$. Then A decreases the distance. Indeed, $1 + 1/(m + n) > 1 + 1/(m^2 + n^2 + 2)$, for all $m \neq n > 0$. It is also clear that A has no fixed points.

Problem 9 (extra credit). Let $f \in C([0, 1])$. Prove that for any $\epsilon > 0$ and $N \in \mathbf{N}$ there exists a function $g \in C([0, 1])$ such that $d_1(f, g) < \epsilon$ and $||g||_2 > N$.
Solution. The idea is based on an observation that a function $x^{-\alpha}, 1/2 \leq \alpha < 1$ is *integrable* but not *square-integrable* on the interval $[0, 1]$. So, we fix $1/2 \leq \alpha < 1$. We also let $M := ||f||_\infty = \max_x |f(x)|$. Next, given $\epsilon > 0$ we can choose $0 < \delta$ such that

$$\left| \int_0^{2\delta} x^{-\alpha} dx \right| = \frac{(2\delta)^{1-\alpha}}{1 - \alpha} < \epsilon/3,$$

as well as $\int_0^{2\delta} |f(x)| dx < \epsilon/3$. We also let $M := ||f||_\infty = \max_x |f(x)|$. In addition, we require $\delta^{1-\alpha} < \epsilon/3$ and $\delta < \epsilon/3M$

We construct $g(x)$ as follows: for $2\delta \leq x \leq 1$, we let $f(x) = g(x)$. For $\delta \leq x \leq 2\delta$, we let $x = (1 + t)\delta, 0 \leq t \leq 1$, and define $g(x) = (1 - t)\delta^{-\alpha} + tf(2\delta)$ (i.e. we interpolate linearly between f and g.

On the interval $[0, \delta]$, we remark that as $\eta \to 0$, we have $\int_\eta^\delta x^{-2\alpha} dx \to \infty$, so we can choose $\eta > 0$ so that $\int_\eta^\delta x^{-2\alpha} dx > N^2$. We finally let $g(x) = x^{-\alpha}$ for $x \in [\eta, \delta]$, and $g(x) = \eta^{-\alpha}$ for $x \in [0, \eta]$. We need to verify that g has the required properties.

For the first property we remark that

$$\int_0^1 |f(x) - g(x)| dx = \int_0^{2\delta} |f(x) - g(x)| dx \leq \int_0^{2\delta} |f(x)| dx + \int_0^{2\delta} |g(x)| dx$$

The first integral in the right-hand side is less than $\epsilon/3$ by the choice of δ. The second integral is less than

$$\int_0^\delta x^{-\alpha} dx + \delta \cdot \max(\delta^{-\alpha}, M) \leq \frac{\epsilon}{3} + \max(\delta^{1-\alpha}, \delta M) < \frac{2\epsilon}{3},$$

also by the choice of δ. Adding the two estimates, we find that $\int_0^1 |f(x) - g(x)| dx < \epsilon$.

For the second inequality, we find that

$$\int_0^1 |g(x)|^2 dx \geq \int_0^\delta |g(x)|^2 dx \geq \int_\eta^\delta x^{-2\alpha} dx > N^2,$$

so $||g||_2 > N$ and the second requirement is satisfied.

Problem 10. Tube Lemma. Let X be a metric space, and let Y be a compact metric space. Consider the product space $X \times Y$. If V is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then V contains some tube $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X. Give an example showing that the Tube Lemma does not hold if Y is not compact.
Solution. Let ρ be the distance on X and σ the distance on Y. We define the *maximum* distance d on $X \times Y$ by

$$d((x_1, y_1), (x_2, y_2)) = \max(\rho(x_1, x_2), \sigma(y_1, y_2)). \tag{7}$$

This defines the d_∞ distance in case $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. It easy to see that open balls for the metric d have the form $U \times V$, where U is an open ball in X, and V is an open ball in Y (and similarly for closed balls). It is also easy to see that the topology defined by the distance $d = \max(\rho, \sigma)$ is equivalent to topologies defined by $d_p := (\rho^p + \sigma^p)^{1/p}$, just like for $\mathbf{R}^2$, (i.e. open and closed sets coincide for all distances), so we can make our calculation using the distance d without loss of generality.

The point (x_0, y) is an interior point of V for all $y \in Y$, hance there exist $r = r(y) > 0$ such that the ball $U_X(x_0, r(y)) \times U_Y(y, r(y))$ centered at (x_0, y) is contained in V. Call the corresponding balls $U_X(y)$ and $U_Y(y)$. The balls $\{U_X(y) \times U_Y(y)\}_{y \in Y}$ form an open cover of $\{x_0\} \times Y$.

Since $\{x_0\} \times Y$ is isometric to Y, it is compact. Accordingly, there exist finitely many $y \in Y$, say $y_1, y_2 \ldots, y_k$ such that $\cup_{j=1}^k U_X(y_j) \times U_Y(y_j)$ cover $\{x_0\} \times Y$. Let $r = \min_{1 \le j \le k}\{r(y_j)\}$. Then we can let $W = U(x_0, r)$ and the conclusion will hold.

For the counterexample in case of noncompact Y, let $X = Y = \mathbf{R}$, $x_0 = 0$ (so that $\{x_0\} \times Y$ is the y-axis), and consider the open set $V = \{(x, y) : |xy| < 1\}$.

Problem 11. Let B denote the set of all sequences (x_n) such that $\lim\limits_{n\to\infty} |x_n| = 0$. Consider l_1 as a subset of l_∞. Prove that the closure of l_1 in l_∞ is equal to B.
Solution. Suppose that $x = (x_1, x_2, ...) \in B$. Consider the sequence $(x^n)_{n=1}^\infty$ of elements in l_1 where $x^n = (x_1, x_2, ..., x_n, 0, 0, ...)$. Then because $\lim\limits_{n\to\infty} |x_n| = 0$ we have $\sup\limits_{i\in\mathbb{N}} |x_i^n - x_i| = d(x^n, x) \to 0$ as $n \to \infty$. So $B \subset cl(l_1)$. To show the reverse inclusion, let $x \in cl(l_1)$ and choose a sequence $(x^n)_{n=1}^\infty$ of elements in l_1 such that $\sup\limits_{i\in\mathbb{N}} |x_i^n - x_i| \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $n \ge N \Rightarrow \sup\limits_{i\in\mathbb{N}} |x_i^n - x_i| \le \frac{\epsilon}{2}$. For the element x^N, choose $M \in \mathbb{N}$ such that $|x_i^N| \le \frac{\epsilon}{2}$ whenever $i \ge M$. Then $\forall i \ge M$, we have (using reverse triangle inequality) $|x_i| - |x_i^N| \le \frac{\epsilon}{2} \Rightarrow |x_i| \le \epsilon$. So $x \in B$.

Problem 12.

(a) Let $A \subset X$ be connected, and let $\{A_\alpha\}_{\alpha \in I}$ be a family of connected subsets of X. Show that $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in I$, then

$$A \cup (\cup_{\alpha \in I} A_\alpha)$$

is connected.

(b) Let X and Y be connected metric spaces. Show that $X \times Y$ is connected.

Solution.
(a) Let $B = A \cup (\cup_{\alpha \in I} A_\alpha)$. Suppose for contradiction that B is not connected. Then by Lemma we can assume that $B \subset C \cup D$, where C and D are disjoint open subsets of X that have nonempty intersection with B. By a result proved in class, we know that $A \cup A_\alpha$ is connected for all α. Then $A \cup A_\alpha$ has to lie entirely in C or entirely in D, otherwise they will separate $A \cup A_\alpha$. Thus $A \cup A_\alpha \subset C$ (say). But this must then hold for *all* α, so $B \subset C$ and $D \cap B = \emptyset$. Contradiction finishes the proof.
(b) Given $x_0 \in X$, consider the map $f : Y \to X \times Y$ given by $f(y) = (x_0, y)$. The map f is continuous and Y is connected, so $\{x_0\} \times Y$ is connected for every Y.

Next, fix $y_0 \in Y$. We similarly find that $X \times \{y_0\}$ is connected. Referring to part (a), let $A = X \times \{y_0\}$, and let $A_x = \{x\} \times Y$, where the index α is replaced by $x \in X$. Now, $A \cap A_x = (x, y_0) \neq \emptyset$. It follows from (a) that

$$(X \times \{y_0\}) \cup (\cup_{x \in X} \{x\} \times Y)$$

is connected. But the above set is just $X \times Y$, so the proof is finished.