McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 5 — due date to be announced

2. The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

   (b) The **Cantor-Lebesgue function** is defined on $\mathcal{C}$ by

   $$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{if } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } b_k = a_k/2.$$

   In this definition, we choose the expansion of x in which $a_k = 0$ or 2. Show that F is well defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ as well as $F(1) = 1$.

   (a) Prove that $F : \mathcal{C} \to [0, 1]$ is surjective, that is, for every $y \in [0, 1]$ there exists $x \in \mathcal{C}$ such that $F(x) = y$.

   (b) One can also extend F to be a continuous function on $[0, 1]$ as follows. Note that if (a, b) is an open interval of the complement of C, then $F(a) = F(b)$. Hence we may define F to have the constant value $F(a)$ in that interval.

9. **Extra-credit.** Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.
   [Hint: Consider the set obtained by taking the union of open intervals which are deleted at the odd steps in the construction of a Cantor-like set.]

14. The purpose of this exercise is to show that covering by a finite number of intervals will not suffice in the definition of the outer measure m_*.
    The **outer Jordan content** $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

    $$J_*(E) = \inf \sum_{j=1}^{N} |I_j|$$

    where the inf is taken over every *finite* covering $E \subset \bigcup_{j=1}^{N} I_j$, by intervals I_j.

    (a) Prove that $J_*(E) = J_*(\overline{E})$ for every set E (here $\overline{E}$ denotes the closure of E).

    (b) Exhibit a countable subset $E \in [0, 1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

16. **The Borel-Cantelli lemma.** Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

    $$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\}$$
$$= \limsup_{k\to\infty}(E_k)$$

(a) Show that E is measurable.

(b) Prove $m(E) = 0$.
[Hint: Write $E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$.]

21. Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.
[Hint: Consider a non-measurable subset of $[0, 1]$, and its inverse image in $\mathcal{C}$ by the function F in Exercise 2.]

27. **Extra credit.** Suppose E_1 and E_2 are a pair of compact sets in $\mathbb{R}^d$ with $E_1 \subset E_2$, and let $a = m(E_1)$ and $b = m(E_2)$. Prove that for any c with $a < c < b$, there is a compact set E with $E_1 \subset E \subset E_2$ and $m(E) = c$.
[Hint: As an example, if $d = 1$ and E is a measurable subset of $[0, 1]$, consider $m(E \cap [0, t])$ as a function of t.]

28. Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$ there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha m_*(I).$$

Loosely speaking, this estimate shows that E contains almost a whole interval.
[Hint: Choose an open set $\mathcal{O}$ that contains E, and such that $m_*(E) \geq \alpha m_*(\mathcal{O})$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.]

29. Suppose E is a measurable subset of R with $m(E) > 0$. Prove that the difference set of E, which is defined by

$$z \in \mathbb{R} : z = x - y \text{ for some } x, y \in E,$$

contains an open interval centered at the origin. If E contains an interval, then the conclusion is straightforward. In general, one may rely on Exercise 28.
[Hint: Indeed, by Exercise 28, there exists an open interval I so that $m(E \cap I) \geq (9/10)m(I)$. If we denote $E \cap I$ by E_0, and suppose that the difference set of E_0 does not contain an open interval around the origin, then for arbitrarily small a the sets E_0, and $E_0 + a$ are disjoint. From the fact that $(E_0 \cup (E_0 + a)) \subset (I \cup (I + a))$ we get a contradiction, since the left-hand side has measure $2m(E_0)$, while the right-hand side has measure only slightly larger than $m(I)$.]

31. **Extra credit.** The result in Exercise 29 provides an alternate proof of the non-measurability of the set $\mathcal{N}$ studied in the text. In fact, we may also prove the non-measurability of a set in $\mathbb{R}$ that is very closely related to the set $\mathcal{N}$.
Given two real numbers x and y, we shall write as before that $x \sim y$ whenever the difference $x - y$ is rational. Let $\mathcal{N}^*$ denote a set that consists of one element in each equivalence class of $\sim$. Prove that $\mathcal{N}^*$ is non-measurable by using the result in Exercise 29.
[Hint: If $\mathcal{N}^*$ is measurable, then so are its translates $\mathcal{N}_n^* = \mathcal{N}^* + r_n$, where $\{r_n\}_{n=1}^{\infty}$ is an enumeration of $\mathbb{Q}$. How does this imply that $m(\mathcal{N}^*) > 0$? Can the difference set of $\mathcal{N}^*$ contain an open interval centered at the origin?]

37. **Extra credit.** Suppose Γ is a curve $y = f(x)$ in $\mathbb{R}^2$, where f is continuous. Show that $m(\Gamma) = 0$.
[Hint: Cover Γ by rectangles, using the uniform continuity of f.]

1. Given an irrational x, one can show (using the pigeon-hole principle, for example) that there exists infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

However, prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3} \quad (\text{or } \leq 1/q^{2+\epsilon})$$

is a set of measure zero.
[Hint: Use the Borel-Cantelli lemma.]