McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 4 — due Friday, October 19

Problem 1 (extra credit). Let $X = C^1[0,1]$ denote the space of continuously differentiable functions on $[0,1]$.

a) Prove that the expression

$$||f||_2 = \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|.$$

defines a norm on X

b) Prove that $(X, ||\cdot||_2)$ is a complete metric space. Is it separable (does it contain a countable dense set)?

c) Prove that $||f||_2$ does not define the same topology on X as the d_∞ norm $\max_{x\in[0,1]} |f(x)|$.

Problem 2. Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of continuously differentiable functions satisfying

$$|f_n(x)| \le M, |f_n'(x)| \le M, \qquad \forall x \in [0,1],\ \forall n \in \mathbf{N}.$$

Prove that $\{f_n\}$ has a uniformly convergent subsequence.

Problem 3. Determine whether the following sets of functions are sequentially compact in $C[0,1]$:

a) $\{(ax)^n\}, n \in \mathbf{N}, a > 0.$

b) $\{\sin(x+n)\}, n \in \mathbf{N}.$

c) $\{e^{x-a}\}, a > 0.$

d) $\{f \in C^2[0,1] : |f(x)| < B_0, |f'(x)| < B_1, |f''(x)| < B_2\}.$

e) (extra credit) $\{f \in C^2[0,1] : |f(x)| < B_0, |f''(x)| < B_2\}.$

f) $\{f \in C^2[0,1] : |f'(x)| < B_1, |f''(x)| < B_2\}.$

Problem 4. Let a_n be a sequence of nonnegative real numbers and let $X = \{x \in l_\infty : |x_n| \le a_n, \forall n\}$. Prove that the following statements are equivalent:

a) X is a compact subset of l_∞.

b) $\lim_{n\to\infty} a_n = 0$.

Problem 5. Let $X \subset l_p$ where $1 \le p < \infty$. Prove that X is compact if and only if the following two conditions hold:

a) X is a closed and bounded subset of l_p.

b) For all $\epsilon > 0$, there exists $n \in \mathbf{N}$ such that $\forall x \in X$ we have $\sum_{n>N} |x_n|^p < \epsilon$.