McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 3 — due Wednesday, October 10

Problem 1. Lipschitz functions. Let M_K be the set of all functions continuous functions on $[0,1]$ satisfying a *Lipschitz condition with constant* $K > 0$, i.e. such that

$$|f(x) - f(y)| \leq K|x - y| \qquad \forall x, y \in [0,1].$$

For $f \in M_K$, define the norm $||f||$ by

$$||f|| = \sup_{x \in [0,1]} |f(x)| + \sup_{x,y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|}.$$

Prove that

i) $||f||$ defines the norm on M_K, i.e. $||c \cdot f|| = |c| \cdot ||f||$ and that $||f + g|| \leq ||f|| + ||g||$.

ii) Conclude that $d(f,g) := ||f - g||$ defines a distance on M_K.

iii) (**extra credit**) M_K is closed, and that it is the closure of the set of all differentiable functions on $[0,1]$ satisfying $|f'(t)| \leq K$.

iv) The set $M = \cup_K M_K$ is not closed.

v) (not for credit). What do you think is the closure of M?

Problem 2. Fredholm equation. Use the fixed point theorem to prove the existence and uniqueness of the solution to *homogeneous Fredholm equation*

$$f(x) = \lambda \int_0^1 K(x,y) f(y) dy.$$

Here $K(x,y)$ is a continuous function on $[0,1]^2$ satisfying

$$|K(x,y)| \leq M$$

is called the *kernel* of the equation. Consider the mapping of $C([0,1])$ into itself given by

$$(Af)(x) = \lambda \int_0^1 K(x,y) f(y) dy.$$

Let $d = d_\infty$ be the usual "maximum" distance between functions. Prove that

i) Prove that $d(Af, Ag) \leq \lambda M \cdot d(f,g)$.

ii) Conclude that A has a unique fixed point in $C([0,1])$ for $|\lambda| < 1/M$, e.g. there exists a unique $f \in C([0,1])$ such that $Af = f$.

iii) Prove that f is a solution of the Fredholm equation.

Problem 3. Relative topology. Let X be a metric space, and let Y be a subset of X (with the induced distance). Prove that a set B is open *in* Y if and only if $B = Y \cap A$, where A is open *in* X.
Problem 4. Let $X = \cup_n X_n$, where X_n is open for all n. Suppose that the restriction $f|X_n$ is continuous for all n; prove that f is continuous on X.
Problem 5. Consider $C([a, b])$, the vector space of all continuous functions on $[a, b]$, equipped with the usual norm $||f||_p, 1 \leq p \leq \infty$. Consider a map $\Phi : C([a, b])C([a, b])$ defined by $\Phi(f) = f^2$. For what values of p is this map continuous? Please justify carefully your answer.
Problem 6. Let M be a bounded subset in $C([0, 1])$. Prove that the set of functions

$$F(x) = \int_0^x f(t)dt, \qquad f \in M$$

has compact closure (in the space of continuous functions with the uniform distance d_∞).
Problem 7. Let X be a compact metric space with a countable base, and let $A : X \to X$ be a map satisfying $d(Ax, Ay) < d(x, y)$ for all $x, y \in X$. Prove that A has a unique fixed point in X.
Problem 8 (extra credit). Give an example of a *non-compact* but *complete* metric space X and a map $A : X \to X$ as in Problem 7 such that A doesn't have a fixed point.
Problem 9 (extra credit). Let $f \in C([0, 1])$. Prove that for any $\epsilon > 0$ and $N \in \mathbf{N}$ there exists a function $g \in C([0, 1])$ such that $d_1(f, g) < \epsilon$ and $||g||_2 > N$.
Problem 10. Tube Lemma. Let X be a metric space, and let Y be a compact metric space. Consider the product space $X \times Y$. If V is an open set of $X \times Y$ containing the slice $\{x_0\} \times Y$ of $X \times Y$, then V contains some tube $W \times Y$ about $\{x_0\} \times Y$, where W is a neighborhood of x_0 in X. Give an example showing that the Tube Lemma does not hold if Y is not compact.
Problem 11. Let B denote the set of all sequences (x_n) such that $\lim_{n\to\infty} |x_n| = 0$. Consider l_1 as a subset of l_∞. Prove that the closure of l_1 in l_∞ is equal to B.
Problem 12.

a) Let $A \subset X$ be connected, and let $\{A_\alpha\}_{\alpha \in I}$ be a family of connected subsets of X. Show that $A \cap A_\alpha \neq \emptyset$ for all $\alpha \in I$, then

$$A \cup (\cup_{\alpha \in I} A_\alpha)$$

is connected.

b) Let X and Y be connected metric spaces. Show that $X \times Y$ is connected.