McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 2 — due Friday, September 28

Problem 1. Let K be the Cantor set. Prove that

a) Prove that the endpoints of the intervals that appear in the construction of K, i.e. the points $0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots$ are everywhere dense in K;

b) **(extra credit).** The numbers of the form $\{t_1 + t_2 : t_1, t_2 \in K\}$ fill the whole interval $[0, 2]$.

Problem 2. Let $A_1, A_2, \ldots$ be subsets of X. We define $\limsup A_n$ by

$$\limsup A_n = \cap_{N=1}^{\infty} \left(\cup_{n \geq N} A_n\right),$$

and $\liminf A_n$ by

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Prove that

a) $x \in \limsup A_n$ iff (if and only if) $x \in A_n$ for infinitely many n.

b) $x \in \liminf A_n$ iff there exists an N such that $x \in A_n, \forall n \geq N$ ($x \in A_n$ with finitely many exceptions).

Problem 3. Let X be a topological space, and let $A \subset X$. Prove that

a) The interior A^{int} of A is the largest open set contained in A;

b) The closure $\overline{A}$ of A is the smallest closed set containing A.

Problem 4. Prove that a subspace A of a complete metric space X is complete if and only if A is closed.

Problem 5. Let (X, d) be a metric space and Y an open subset of X. Show that if A is dense in X, then $A \cap Y$ is dense in Y. Give a counterexample to the previous statement when Y is closed.

Problem 6, Jensen's inequality (extra credit). Let $f(x)$ be Riemann integrable on $[a, b]$; $f(x) \in [c, d], \forall x \in [a, b]$; and that $\phi(y)$ is convex on $[c, d]$. Prove that

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \leq \frac{1}{b-a}\int_a^b \phi(f(x))dx.$$

For simplicity, you can assume that f is continuous.

Problem 7. Let $a < b, b - a = 1$, and let $f \in C([a, b])$.

a) Prove that

$$p \to ||f||_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$$

is non-decreasing for $p \geq 1$. Hint: use the Hölder's inequality for functions.

b) In a), find $\lim_{p\to\infty} ||f||_p$.

c) Does the statement in a) hold if $b - a \neq 1$? Justify your answer.

Problem 8 (extra credit). Let $f \in C([a,b]), f \not\equiv 0$. For $p \geq 1$, define

$$\phi(p) := ||f||_p^p = \int_a^b |f(x)|^p dx.$$

a) If $1 \leq r < p < s < \infty$, prove that

$$||f||_p \leq \max(||f||_r, ||f||_s).$$

b) Prove that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

c) Prove that $\log \phi(p)$ is convex.

Problem 9 (extra credit). Give an example of a complete metric space X and a sequence of *nested* closed balls (i.e. $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots \supseteq B_n \supseteq \ldots$) such that

$$\cap_{n=1}^{\infty} B_n = \emptyset.$$

Hint: the radius of the balls need not go to zero.

Problem 10 (extra credit).

A sequence $\{x_n\}$ in a metric space (X,d) is called a *Cauchy sequence* if $\limsup_{k\geq 1} d(x_n, x_{n+k})$ goes to zero as $n \to \infty$, or, equivalently, if for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $d(x_n, x_{n+k}) < \epsilon$ for any $n \geq N$ and $k \geq 0$.

Prove that the following sequences of rational numbers are Cauchy sequences with respect to the p-adic distance d_p introduced in Assignment 1:

a) $x_n = 1 + p + p^2 + \ldots + p^n$;

b) $y_n = (p-1) + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^n$.

Find the limits (with respect to the p-adic distance d_p) of the following sequences:

c) $(1-p)x_n$, where x_n is as in a);

d) $1 + y_n$, where y_n is as in b).