McGill University

Math 354: Honors Analysis 3 — Fall 2012

Assignment 1 — due Monday, September 24

Problem 1.

(i) Verify the identity

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - b_i a_j)^2.$$

ii) Let $f(x)$ and $g(x)$ be continuous functions on $[a, b]$. Prove that

$$\left(\int_a^b f(x)g(x)dx\right)^2 = \int_a^b (f(x))^2 dx \cdot \int_a^b (g(x))^2 dx - \frac{1}{2}\int_a^b \int_a^b \left[f(x)g(y) - g(x)f(y)\right]^2 dxdy.$$

Problem 2.

(i) Starting from the inequality $xy \le x^p/p + y^q/q$, where $x, y, p, q > 0$ and $1/p + 1/q = 1$, deduce *Hölder's integral inequality* for continuous functions $f(t), g(t)$ on $[a, b]$:

$$\int_a^b f(t)g(t)dt \le \left(\int_a^b |f(t)|^p dt\right)^{1/p} \left(\int_a^b |g(t)|^q dt\right)^{1/q};$$

(ii) Use (i) to prove *Minkowski's integral inequality* for continuous functions $f(t), g(t)$ on $[a, b]$ and $p \ge 1$:

$$\left(\int_a^b |f(t) + g(t)|^p dt\right)^{1/p} \le \left(\int_a^b |f(t)|^p dt\right)^{1/p} + \left(\int_a^b |g(t)|^p dt\right)^{1/p}.$$

Problem 3. Prove that the set of all points $x = (x_1, x_2, \dots, x_k, \dots)$ with only finitely many nonzero coordinates, each of which is a rational number, is dense in the space l^2 of sequences.

Problem 4 (extra credit).

i) Suppose $\phi \in C([a, b])$ (which need not be differentiable) satisfies

$$\phi((x + y)/2) \le (\phi(x) + \phi(y))/2, \qquad x, y \in [a, b].$$

Prove that for all $x, y \in [a, b]$, and for any $t \in [0, 1]$, we have

$$\phi(tx + (1 - t)y) \le t\phi(x) + (1 - t)\phi(y), \tag{1}$$

i.e. that ϕ is *convex* on $[a, b]$.

ii) Assume that a function ϕ (that is *not* assumed to be continuous on $[a, b]$), satisfies (1). Prove that ϕ is then actually continuous on $[a, b]$.

iii) Prove that if $\phi \in C^2([a, b])$, and $\phi''(x) > 0, \forall x \in [a, b]$, then ϕ is convex on $[a, b]$.

iv) Prove that if $x_1, \ldots, x_n \in [a,b]$, and $t_1, \ldots, t_n > 0$ satisfy $t_1 + \ldots + t_n = 1$, and if ϕ is convex on $[a,b]$, then
$$\phi(t_1x_1 + \ldots + t_nx_n) \leq t_1\phi(x_1) + \ldots + t_n\phi(x_n).$$

Problem 5. Let X be a metric space, $A \subseteq X$ a subset of X, and x a point in X. The *distance from x to A* is denoted by $d(x, A)$ and is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Prove that

i) If $x \in A$, then $d(x, A) = 0$, but not conversely;

ii) For a fixed A, $d(x, A)$ is a continuous function of x;

iii) $d(x, A) = 0$ if and only if x is a contact point of A (i.e. every neighborhood of x contains a point from A);

iv) The closure $\overline{A}$ satisfies
$$\overline{A} = A \cup \{x : d(x, A) = 0\}.$$

Problem 6. Let (X, d) be a metric space, and $f : X \to \mathbf{R}$ a continuous function. The *nodal set* of f, denoted by $Z(f)$, is the set $\{x \in X : f(x) = 0\}$.

i) Prove that $Z(f)$ is a closed subset of X.

Next, let A, B be two closed nonempty subsets of X, $A \cap B = \emptyset$. Let $d(x, A)$ (resp. $d(x, B)$) denote the distance from $x \in X$ to A (resp. B), defined in Problem 5 in Assignment 1. Define a function $F : X \to \mathbf{R}$ by the formula

$$F(x) \quad = \quad \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Prove that

ii) F is continuous;

iii) $F(x) = 0$ iff $x \in A$, and $F(x) = 1$ iff $x \in B$.

Problem 7. Let Mat_n denote the space of $n \times n$ real matrices. For $A \in \mathrm{Mat}_n$, define the norms $||A||_1$ as follows:
$$||A||_1 = \sup_{0 \neq \mathbf{x} \in \mathbf{R}^n} \frac{||A\mathbf{x}||}{||\mathbf{x}||},$$
where $||x||$ is the usual Euclidean norm. Next define another norm $||A||_2$ by

$$||A||_2 = \max_{i,j} |A_{ij}|.$$

Prove that

i) Prove that $||A||_{1,2}$ defines a norm on Mat_n;

ii) Prove that there exists a constant $C_n > 1$ such that $1/C_n \leq ||A||_1/||A||_2 \leq C_n$.

Problem 8 (extra credit). Let p be a prime number (a positive integer that is only divisible by 1 and itself, e.g. $p = 2, 3, 5, 7, 11$ etc). Define *p-adic distance* d_p on the set $\mathbf{Q}$ of rational numbers as follows: given $q_1 q_2 \in \mathbf{Q}$, let $|q_1 - q_2| = q \in \mathbf{Q}$. If $q_1 = q_2, q = 0$, then we set $d_p(q_1, q_2) = 0$. If $q \neq 0$, we can write q as

$$q = p^m \, \frac{a}{b}, \qquad \textit{where } m \in \mathbf{Z},\ GCD(a,b) = 1,\ GCD(a,p) = GCD(b,p) = 1.$$

Here $GCD(a, b)$ is the greatest common divisor of two natural numbers a and b. Then we define the p-adic distance by

$$d_p(q_1, q_2) = p^{-m}.$$

Please, note the minus sign in the definition.

Examples: $d_2(5/2, 1/2) = 1/2$; $d_3(17, 8) = 1/9$; $d_5(4/15, 1/15) = 5$.

Prove that d_p satisfies all the properties of a distance. The only nontrivial part is the triangle inequality:

$$d_p(q_1, q_2) + d_p(q_2, q_3) \geq d_p(q_1, q_3).$$

You may use without proof all standard properties of the greatest common divisor, prime decomposition etc.

Problem 9 (extra credit).

Denote by $\mathcal{P}$ the set of polygons in $\mathbf{R}^2$, not necessarily convex. A *polygon* P with vertices $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is the set of points in $\mathbf{R}^2$ bounded by a simple closed curve that is a union of line segments

$$[\mathbf{x}_1, \mathbf{x}_2], [\mathbf{x}_2, \mathbf{x}_3], \ldots, [\mathbf{x}_{n-1}, \mathbf{x}_n], [\mathbf{x}_n, \mathbf{x}_1].$$

The boundary curve is denoted ∂P and is sometimes called a *polyline* or a *broken line*. We require that different line segments do not intersect except at common endpoints.

A *symmetric difference* of two sets A, B is denoted by $A \Delta B$ and is defined by

$$A \Delta B = (A \backslash B) \cup (B \backslash A),$$

where $A \backslash B = A \cap B^c$ is the set of points $\{x \in A, x \notin B\}$.

Given two polygons $P_1, P_2 \in \mathbf{R}^2$, define the distance between them by

$$d(P_1, P_2) \ = \ \text{Area}(P_1 \Delta P_2).$$

Prove that d satisfies all the properties of a distance. Hint: if $X \subset Y$, then $\text{Area}(X) \leq \text{Area}(Y)$.