McGill University
Math 354: Honors Analysis 3

TAKE HOME MIDTERM 2012 — Solutions

Do any 8 of the following 10 problems. Every problem are worth 10 points.
Problem 1.

a) Let X be a metric space with the distance d_1. Prove that $d_2(x,y) = d_1(x,y)/(1+d_1(x,y))$ also defines the distance on X. Prove that open sets and Cauchy sequences for d_1 and d_2 coincide.

b) Prove the same results for $d_3(x,y) = \min\{d_1(x,y), 1\}$.

c) Define the distance on the set X of all sequences $x = (x_1, x_2, \ldots)$ of real numbers by the formula
$$d(x,y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}.$$
Prove that d defines a distance on X, and that X is complete with respect to d. Is X separable?

Solution: See solutions to problems in Math 564, Assignment 5, posted on the course page.
Problem 2. Let $X = \Pi_i X_i$ be a product of metric spaces, with the distance d_i. Consider the product topology on X (a basis of open sets is given by $\Pi_i U_i$, where $U_i = X_i$ except for finitely many i-s). Let $\rho_i = d_i/(1+d_i)$; it preserves the topology of X_i by Problem 1a. Prove that
$$\rho(x,y) = \sum_{j=1}^{\infty} \frac{\rho_j(x_j, y_j)}{2^j}$$
defines a distance on X, and that the topology given by ρ coincides with the product topology. Hint: Let U be an open set in the basis for the product topology, and let $x \in U$. Prove that there exists $r > 0$ s.t. $B_\rho(x,r) \subset U$. Conversely, let $y \in B_\rho(x,r)$. Prove that there exists a basis set U for the product topology s.t. $y \in U \subset B_\rho(x,r)$.
Solution:

The proof that ρ defines the distance is routine. We focus on the second part of the problem, that the distance induces the product topology on $\Pi_j X_j$.

By Problem 1, we know that d_j and $\rho_j = d_j/(1+d_j)$ determine the same topology on X_j. We know that ρ_j satisfies $\rho_j \le 1$; the distance on the product is defined by
$$\rho(x,y) = \sum_{j=1}^{\infty} \frac{\rho_j(x_j, y_j)}{2^j}.$$

Let
$$U = U_1 \times U_2 \times \ldots \times U_m \times X_{m+1} \times \ldots$$
be a cylinder in the product topology, and let $x = (x_1, x_2, \ldots) \in U$. Since each U_j is open, there exist $r_j, 1 \le j \le m$ such that $B(x_j, r_j) \subset U_j$, where the ball is taken with respect to the distance ρ_j. We want to find r such that
$$B_\rho(x,r) \subset B_{\rho_j}(x_j, r_j), \qquad 1 \le j \le m. \tag{1}$$

It will then follow that $B_\rho(x,r) \subset U$, proving the first part of the required statement.

Now, we claim it suffices to take
$$r < \min_{1\le j\le m} r_j/2^j.$$
for (1) to hold. Indeed, if $\rho(x,y) < r$, then $\rho_j(x_j,y_j)/2^j < r$, and it follows that $\rho_j(x_j,y_j) < 2^j \cdot r < r_j$, as required.

In the other direction, let $y \in B_\rho(x,r)$. Let $rho(x,y) = r_1 < r$. Let m be such that $2^{-m} < (r-r_1)/4$. We shall choose a set $U = U_1 \times \ldots \times U_m$, where $U_j = B_{\rho_j}(y_j, r_j)$. It is clear that one can choose r_j-s so that
$$\sum_{j=1}^m \frac{r_j}{2^j} < \frac{r-r_1}{4}; \tag{2}$$
indeed, it suffices to take $r_j < (r-r_1)/(4m)$. Assume now that $z \in U_1 \times \ldots \times U_m \times X_{m+1} \times \ldots$ Then
$$\rho(y,z) = \sum_j \frac{\rho_j(x_j,y_j)}{2^j} < \sum_{j=1}^m r_j 2^j + \sum_{j=m+1}^\infty \frac{1}{2^j} < \frac{r-r_1}{4} + \frac{1}{2^m} < \frac{r-r_1}{2},$$
where we have used the inequality (2) and our choice of m. It follows that $\rho(z,x) \le \rho(z,y) + \rho(y,x) < r_1 + (r-r_1)/2 < r$, and so $U \subset B_\rho(x,r)$ as required, QED.

Problem 3. Let $C^\infty[a,b]$ denote the space of infinitely differentiable functions on $[a,b]$ (all the derivatives exist and are continuous). Let
$$d(f,g) = \sum_{k=1}^\infty \frac{\max_{x\in[a,b]} |f^{(k)}(x) - g^{(k)}(x)|}{2^k(1 + \max_{x\in[a,b]} |f^{(k)}(x) - g^{(k)}(x)|)}.$$
Prove that d defines the distance on $C^\infty[a,b]$, and that the resulting metric space is complete.

Solution: See solutions to problems in Math 564, Assignment 5, posted on the course page.

Problem 4. Hausdorff distance. Let $A, B \subset X$, where X is a metric space. Let $U_r(Z) := \{x \in X : d(x,Z) \le r\}$. Define the *Hausdorff distance*
$$d_H(A,B) = \inf\{r > 0 : A \subset U_r(B),\ \textit{and}\ B \subset U_r(A)\}.$$

a) Prove that $d_H \ge 0$, is symmetric and satisfies the triangle inequality.

b) Prove that $d_H = \max(\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A))$.

c) Prove that $d_H(A,B) \le r$ if and only if $d(a,B) \le r$, for all $a \in A$ and $d(b,A) \le r$ for all $b \in B$.

d) $d_H(A,\overline{A}) = 0$, where $\overline{A}$ is the closure of A.

e) Show that if A, B are closed subsets of X, and $d_H(A,B) = 0$, then $A = B$.

According to e), d_H defines a distance on the set $\mathcal{M}(X)$ of all *closed* subsets of X.

Solution: See the paper by J. Henrikson on Hausdorff distance
`http://www-math.mit.edu/phase2/UJM/vol1/HAUSF.PDF`

Problem 5. Let $A_n \in \mathcal{M}(X)$ be a sequence of closed subsets of X, and let $d_H(A_n, A) \to 0$ as $n \to \infty$, i.e. let $A_n \to A$ in the metric space $(\mathcal{M}(X), d_H)$. Prove that

a) A is the set of limits of all converging subsequences $\{a_n\}$ in X, s.t. $a_n \in A_n$ for all n.

b) $A = \cap_{n=1}^\infty(\textit{closure of}\ \cup_{m=n}^\infty A_m)$.

Next, let X be compact, and $\{A_i\}$ be a sequence of its compact subspaces. Prove that

c) If $A_{i+1} \subset A_i$ for all i, then $A_k \to \cap_{k=1}^{\infty} A_k$ in $\mathcal{M}(X)$.

d) If $A_i \subset A_{i+1}$ for all i, then A_i converges to the closure of $\cup_{i=1}^{\infty} A_i$.

Solution:
See the paper by J. Henrikson on Hausdorff distance
`http://www-math.mit.edu/phase2/UJM/vol1/HAUSF.PDF`

Problem 6. Consider the orthogonal group $O(n)$ consisting of all $n \times n$ orthogonal matrices, i.e. matrices whose columns form an orthonormal basis $v_1, \dots, v_n$ of $\mathbf{R}^n$. We introduce the topology on $O(n)$ by considering it as a subspace of $\mathbf{R}^{n^2}$ (consider the matrix entries as coordinates). Show that

a) $O(n)$ is a closed subset of $\mathrm{Mat}_n(\mathbf{R}) \simeq \mathbf{R}^{n^2}$, by considering the dot products (v_i, v_j) of the columns of matrices in $O(n)$ as functions from $\mathrm{Mat}_n(\mathbf{R})$ into $\mathbf{R}$.

b) Show that $O(n)$ is compact.

c) Prove that $O(n)$ is a group, i.e. if $A, B \in O(n)$, then $AB \in O(n)$, and $A^{-1} \in O(n)$.

Solution, sketch: The mapping $\pi_{ij} : M \to (v_i, v_j)$ is a continuous function for all (i, j), so

$$O(n) = \left(\cap_{i \neq j} \pi_{ij}^{-1}(\{0\})\right) \cap \left(\cap_i \pi_{ii}^{-1}(\{1\})\right)$$

is an intersection of closed sets and hence is closed. Is is also clearly bounded, hence compact by a characterization of compact subsets of $\mathbf{R}^{n^2}$.

Problem 7. Let $X = C^m[0,1]$ denote the space of m times continuously differentiable functions on $[0,1]$. Define the norm on X by

$$||f|| = \sum_{k=0}^{m} \max_{x \in [0,1]} |f^{(k)}(x)|.$$

Prove that $(X, ||\cdot||)$ is a complete metric space. Is it separable?

Solution: See solutions to problems in Math 564, Assignment 5, posted on the course page.

Problem 8.

a) Compute the area $A(r)$ of the ball of radius r in $\mathbf{R}^2, S^2$, and $\mathbf{H}^2$. Hint: the volume element in polar coordinates (r, θ) is given by $r dr d\theta$ in $\mathbf{R}^2$; $\sin r dr d\theta$ on S^2; and $\sinh r dr d\theta$ in $\mathbf{H}^2$. Where does the volume grow faster? Compute the first 3 terms in the Taylor series expansion of the volume as $r \to 0$; what do you get?

b) Next, compute he length $L(r)$ of the circle of radius r in $\mathbf{R}^2, S^2$, and $\mathbf{H}^2$. Hint: the length element in polar coordinates (r, θ) is given by $dr^2 + r^2 d\theta^2$ in $\mathbf{R}^2$; $dr^2 + \sin^2 r d\theta^2$ on S^2; and $dr^2 + \sinh^2 r d\theta^2$ in $\mathbf{H}^2$.

c) Describe the behavior of the ratio $A(r)/L(r)$ as $r \to 0$.

d) Describe the behavior of the ratio $L(r)/A(r)$ as $r \to \infty$ in $\mathbf{R}^2$ and $\mathbf{H}^2$; and as $r \to \pi$ in S^2.

Solution: Explained in class.

Problem 9.

a) Compute $L(r)$ and $A(r)$ on an infinite k-regular tree, $k \geq 2$. Describe the behavior of the ratio $L(r)/A(r)$ as $r \to \infty$.

b) Do the same for the graph $\mathbf{Z}^2$.

Solution: Similar to Problem 8, it is a discrete analogue.
Problem 10. Every real number in $x \in [0,1]$ can be expanded into a (finite or infinite) *continued fraction*

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}},$$

sometimes denoted by $x = [n_1, n_2, n_3, \dots]$.

a) Prove that finite continued fractions correspond to rational numbers, while infinite fractions correspond to irrational numbers.

c) Describe all the *periodic* continued fractions, $x = [n_1, \dots, n_k, n_1, \dots, n_k, \dots]$.

Solution: Let f denote the map $f(x) = \{1/x\}$, where $\{y\}$ denotes the fractional part of y. If

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}, \tag{3}$$

then the function f can be written as a *shift map*,

$$f([n_1, n_2, n_3, \dots]) = [n_2, n_3, \dots].$$

since

$$1/x = n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}.$$

Clearly, finite continued fractions give rise to rational numbers (clear the denominators). Conversely, if we apply the map f to a rational number $p/q, 0 < p < q$, the result $\{q/p\}$ will have a smaller denominator p, so after $\leq q$ applications of f we shall get 1 and the continued fraction will terminate.

Every periodic real number x satisfies the equation

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \dots + \frac{1}{n_k + x}}}.$$

Clearing the denominators, it is easy to see by induction that x satisfies

$$x = \frac{Ax + B}{Cx + D}$$

where A, B, C, D are integers that depend on $n_1, \dots, n_k$; it follows that x satisfies quadratic equation with integer coefficients, so it is a *quadratic irrational.* In fact, every quadratic irrational gives rise to *eventually periodic* continued fraction, but we won't prove it.