Math 320a, Differential Geometry, D. Jakobson, Fall 2003

Proof of the THEOREMA EGREGIUM of Gauss

Let $X(u,v)$ be a coordinate patch on our surface, and let N be the unit normal vector. We first express various partial derivatives of X, N in the basis X_u, X_v, N of $\mathbf{R}^3$; we take into account the fact that $e = X_{uu} \cdot N, g = X_{vv} \cdot N, f = X_{uv} \cdot N = X_{vu} \cdot N$:

$$\begin{aligned} X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + eN, \\ X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + gN, \\ X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + fN, \\ X_{vu} &= \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + fN, \\ N_u &= a_{11} X_u + a_{21} X_v, \\ N_v &= a_{12} X_u + a_{22} X_v. \end{aligned} \tag{1}$$

Here a_{ij} are given by

$$(a_{11}, a_{12}, a_{21}, a_{22}) = (fF - eG, gF - fG, eF - fE, fF - gE)/(EG - F^2). \tag{2}$$

The coefficients Γ^k_{ij} are called *Christoffel symbols*. Since $X_{uv} = X_{vu}$ we see immdediately that $\Gamma^i_{12} = \Gamma^i_{21}$ for $i = 1, 2$.

We next prove the following

Lemma 1. *The following identities hold: (1) $X_{uu} \cdot X_u = E_u/2$; (2) $X_{uu} \cdot X_v = F_u - E_v/2$; (3) $X_{uv} \cdot X_u = E_v/2$; (4) $X_{uv} \cdot X_v = G_u/2$; (5) $X_{vv} \cdot X_u = F_v - G_u/2$; (6) $X_{vv} \cdot X_v = G_v/2$.*

Proof. **(1):** Differentiate $E = X_u \cdot X_u$ with respect to u; **(6):** Switch u and v in (1). **(3):** Differentiate $E = X_u \cdot X_u$ with respect to v. **(4):** Switch u and v in (3). **(2):** Use $dF/du = X_{uu} \cdot X_v + X_u \cdot X_{uv}$ and (3). **(5):** Switch u and v in (2). □

Taking inner products of the first equation in (1) with X_u and X_v we obtain (using the identities (1) and (2) of Lemma 1)

$$\begin{aligned} \Gamma^1_{11} E + \Gamma^2_{11} F &= X_{uu} \cdot X_u = E_u/2, \\ \Gamma^1_{11} F + \Gamma^2_{11} G &= X_{uu} \cdot X_v = F_u - E_v/2. \end{aligned} \tag{3}$$

Taking inner products of the second equation in (1) with X_u and X_v we obtain (using the identities (5) and (6) of Lemma 1)

$$\begin{aligned} \Gamma^1_{22} E + \Gamma^2_{22} F &= X_{vv} \cdot X_u = F_v - G_u/2, \\ \Gamma^1_{22} F + \Gamma^2_{22} G &= X_{vv} \cdot X_v = G_v/2. \end{aligned} \tag{4}$$

Taking inner products of the third equation in (1) with X_u and X_v we obtain (using the identities (3) and (4) of Lemma 1)

$$\begin{aligned} \Gamma^1_{12} E + \Gamma^2_{12} F &= X_{uv} \cdot X_u = E_v/2, \\ \Gamma^1_{12} F + \Gamma^2_{12} G &= X_{uv} \cdot X_v = G_u/2. \end{aligned} \tag{5}$$

The formulas (3), (4), (5) are systems of linear equations for Γ^k_{ij}; the determinant for all systems is $EG - F^2 \neq 0$, so we have proved

Proposition 2. *The coefficients Γ^k_{ij} can be expressed in terms of E, F, G and their derivatives.*

We now use the identity $(X_{uu})_v = (X_{uv})_u$. We substitute the first (respectively, the third) equations of (1) for X_{uu} (respectively, X_{uv}) and obtain the following equality:

$$\begin{aligned} &\Gamma_{11}^1 X_{uv} + \Gamma_{11}^2 X_{vv} + eN_v + (\Gamma_{11}^1)_v X_u + (\Gamma_{11}^2)_v X_v + e_v N = \\ &\Gamma_{12}^1 X_{uu} + \Gamma_{12}^2 X_{vu} + fN_u + (\Gamma_{12}^1)_u X_u + (\Gamma_{12}^2)_u X_v + f_u N \end{aligned} \tag{6}$$

Since X_u, X_v, N are linearly independent, the coefficients of X_u, X_v, N should be zero after we collect the terms in (6).

We collect the coefficients of X_v in (6) and use (1) once again to obtain

$$\Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 + ea_{22} + (\Gamma_{11}^2)_v = \Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 + fa_{21} + (\Gamma_{12}^2)_u. \tag{7}$$

Substituting the values of e, f (cf. (2)) into (7), we find that

$$\begin{aligned} &\Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 + (\Gamma_{11}^2)_v - \left[\Gamma_{12}^1 \Gamma_{11}^2 + \Gamma_{12}^2 \Gamma_{12}^2 + (\Gamma_{12}^2)_u\right] = \\ &\frac{f(eF - fE) - e(fF - gE)}{EG - F^2} = E\frac{eg - f^2}{EG - F^2} = EK, \end{aligned} \tag{8}$$

where K is the Gauss curvature. Proposition 2 and (8) together finish the proof of the THEOREMA EGREGIUM of Gauss. □