Math 256, D. Jakobson

A Review pf Determinants

A determinant of an $n \times n$ matrix A is a *multilinear* function of its rows $A_1, \ldots, A_n$ and of its columns $A^1, \ldots, A^n$, i.e. for any $1 \leq i \leq n$

$$\begin{aligned} &\det(A_1, \ldots, A_{i-1}, \lambda B_1 + \mu B_2, A_{i+1}, \ldots, A_n) = \\ &\quad \lambda \cdot \det(A_1, \ldots, A_{i-1}, B_1, A_{i+1}, \ldots, A_n) + \\ &\quad \mu \cdot \det(A_1, \ldots, A_{i-1}, B_2, A_{i+1}, \ldots, A_n) \end{aligned}$$

and a similar identity holds for columns. It is also an *alternating* function, i.e.

$$\det(\ldots, A_i, \ldots, A_j, \ldots) = -\det(\ldots, A_j, \ldots, A_i, \ldots)$$

or, equivalently, determinant is equal to 0 whenever two rows (or columns) of the matrix are the same. The determinant is a unique such function satisfying $\det(\mathrm{Id}_n) = 1$.

Suppose vectors $w_1, \ldots, w_n$ are linear combinations of $v_1, \ldots, v_n$ with coefficients

$$\begin{cases} w_1 &= a_{11} v_1 + \ldots + a_{1n} v_n, \\ & \ldots \\ w_n &= a_{n1} v_1 + \ldots + a_{nn} v_n. \end{cases}$$

Let the i-th row of the matrix A be equal to v_i, and let the i-th row of B equal to w_i. Then

$$\det B = \det A \cdot \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1,\sigma(1)} \ldots a_{n,\sigma(n)}$$

where $\varepsilon(\sigma)$ is the sign of a permutation σ.

Another useful formula for computing determinats is the *Laplace expansion*:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where a_{ij} are the elements fo the i-th row of A, and A_{ij} denotes the matrix A with the i-th row and the j-th column removed. An analogous formula is true for columns.

The determinant satisfies $det(A) = \det({}^tA)$ and $\det(AB) = \det(A)\det(B)$. If C is invertible, then $\det(C^{-1}AC) = \det(A)$ (so the determinant of a matrix corresponding to a linear transformation doesn't depend on a choice of a basis). Suppose that the columns vectors $A^1, \ldots, A^n, B$ satisfy

$$B = \sum_{i=1}^{n} x_i A^i$$

where x_i-s are constants. Then the following identity (called *Cramer's rule*) holds for all $1 \leq i \leq n$:

$$x_i \det(A^1, \ldots, A^i, \ldots, A^n) = \det(A^1, \ldots, B, \ldots, A^n)$$

(the i-th column is replaced by B).

The following elementary operations row are useful when computing determinants:

- Interchanging two rows (det changes sign).

- Multiplying a row by $\lambda \neq 0$ (det gets multiplied by λ).
- Adding to a row of a matrix a linear combination of *other* rows of that matrix (det doesn't change).

Similar operations for columns can be used as well (note, though, that you can only use row operations when solving a system of equations or computing an inverse of a matrix).

The matrix (over a field) is invertible iff its determinant is not equal to zero iff its row/columns are linearly independent iff its rank is equal to its size.

The *adjoint* (or *adjugate*) $\tilde{A} = (b_{ij})$ of a matrix A is defined by $b_{ij} = (-1)^{i+j} \det(A_{ji})$. This matrix satisfies

$$\tilde{A}A \;=\; A\tilde{A} \;=\; \det(A) \cdot \mathrm{Id}.$$