McGill University — Math 315A: Differential Equations

MIDTERM EXAMINATION — SOLUTIONS

**Problem 1.** Find the general solution of the equation

$$2x^2y' = x^2 + y^2, \qquad x > 0.$$

**Solution:** This equation can be rewritten as

$$y' = (x^2 + y^2)/(2x^2) = 1/2 + (y/x)^2/2$$

ans so is a homogeneous equation. Let $u = y/x$. Then $y' = (ux)' = u'x + u = (1 + u^2)/2$ or $u'x = (u - 1)^2/2$ which is a separable equation, and so

$$\int \frac{dx}{2x} = \int \frac{du}{(u-1)^2},$$

so

$$\ln x = (-2)/(u - 1) + C = 2x/(x - y) + C$$

and therefore

$$2x = (x - y)(\ln x - C).$$

or

$$y = \frac{x(2 + C - \ln x)}{C - \ln x}$$

**Problem 2.** Solve the equation

$$[3 - \sin(xy)]dx + [2 + 3x/y - x\sin(xy)/y]dy = 0, \qquad y(0) = 1. \tag{1}$$

**Solution:** Let $M(x, y) = 3 - \sin(xy)$, $N(x, y) = 2 + 3x/y - x\sin(xy)/y$. Then

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x\cos(xy) - 3/y + \sin(xy)/y + x\cos(xy) = (\sin(xy) - 3)/y \neq 0,$$

so the equation (1) is not exact. However,

$$\frac{\partial M/\partial y - \partial N/\partial x}{-M} = \frac{\sin(xy) - 3}{y(\sin(xy) - 3)} = \frac{1}{y}$$

is a function depending on $y$ only, so the equation (1) has an integrating factor $\mu$ that is a function of $y$. That function is given by the formula

$$\mu(y) = \exp\left(\int \frac{dy}{y}\right) = y.$$

After a multiplication by $\mu(y) = y$, the equation becomes

$$(3y - y\sin(xy))dx + (2y + 3x - x\sin(xy))dy = 0 \tag{2}$$

To solve the exact equation (2), we have to find a function $F = F(x, y)$ satisfying

$$\begin{cases} \partial F/\partial x = 3y - y\sin(xy), \\ \partial F/\partial y = 2y + 3x - x\sin(xy). \end{cases} \tag{3}$$

Integrating the first equation in (3), we find that

$$F(x, y) = \int (3y - y\sin(xy))dx = 3xy + \cos(xy) + f(y).$$

Substituting into the second equation in (3), we get

$$\partial F/\partial y = 3x - x\sin(xy) + f'(y) = 2y + 3x - x\sin(xy).$$

It follows that $f(y) = \int 2ydy = y^2$. We conclude that soltuions of (1) are given implicitly by

$$F(x, y) = 3xy + \cos(xy) + y^2 = C.$$

Substituting the initial condition $x = 0, y = 1$ we find that $C = 2$, so the solution is given implicitly by

$$3xy + \cos(xy) + y^2 = 2.$$

**Problem 3.** Find the general solution of the differential equation

$$y'' + 2y' - 3y = 9x. \tag{4}$$

**Solution:** The quadratic equation associated to the homogeneous differential equation $y'' + 2y' - 3y = 0$ is $\lambda^2 + 2\lambda - 3 = 0$. It has roots $\lambda = -3$ and $\lambda = 1$, so a general solution of the homogeneous equation is given by $y = ce^{-3x} + de^x$. We want to find a particular solution $y_p$ of the nonhomogeneous equation (4) by the method of undetermined coefficients. The trial solution has the form

$$y_p(x) = x^s(ax + b) \qquad s = 0, 1 \textit{ or } 2.$$

We start with

$$y_p(x) = ax + b.$$

Neither $x$ nor 1 are solutions of the homogeneous equation, so there is no need to multiply by $x$. Substituting $y_p$ in (4) we find that $y_p'' + 2y_p' - 3y_p = (2a - 3b) - 3ax = 9x$.

Equating the coefficients, we find that $-3a = 9$ and $2a - 3b = 0$. Accordingly, $a = -3$ and $b = -2$, and

$$y_p(x) = -3x - 2.$$

To find $y_{gen}$ we add the general solution of the homogeneous equation:

$$y_{gen}(x) = -3x - 2 + ce^{-3x} + de^{x}.$$

**Problem 4.** Find a particular solution of the differential equation

$$y'' + 9y = 6\sin(3x). \tag{5}$$

**Solution:**

The solution of the homogeneous equation $y'' + 4y = 0$ is given by $y = c_1\cos(3x) + c_2\sin(3x)$. We want to find a particular solution $y_p$ of the nonhomogeneous equation (4) by the method of undetermined coefficients. A trial solution has the form

$$y_p(x) = x^s(a\sin(3x) + b\cos(3x)).$$

where $s = 0, 1$ or 2.

For $s = 0$, we find that both $\sin(3x)$ and $\cos(3x)$ are solutions of of the homogeneous equation, so we have to multiply by $x$. After mutliplication, we get $y_p(x) = ax\sin(3x) + bx\cos(3x)$. Neither $x\sin(3x)$ nor $x\cos(3x)$ are solutions of the homogeneous equation, so we should look for a solution of (5) in the form

$$y_p(x) = ax\sin(3x) + bx\cos(3x). \tag{6}$$

We substitute (6) into (5) to find $a$ and $b$. We get $y_p'' + 9y_p = 6a\cos(3x) - 6b\sin(3x) = 6\sin(3x)$. Accordingly, we find that $a = 0, b = -1$, and a particular solution of (5) is given by

$$y_p(x) = -x\cos(3x).$$