This is a summary of various results about solving constant coefficients heat equation on the interval, both homogeneous and inhomogeneous.

1. Homogeneous equation

We only give a summary of the methods in this case; for details, please look at the notes Prof. Xu or L. Laayouni.

1.1. **Zero BC.**

$$u_t = \beta u_{xx}, \quad u(x,0) = f(x), \quad u(0,t) = 0 = u(L,t).$$

To solve, expand $f(x)$ in *sine* Fourier series; this corresponds to the *odd* extension of $f(x)$ to the interval $[-L, L]$, e.g. $f(-x) = -f(x)$. The expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right), \qquad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx.$$

The solution $u(x,t)$ is then given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \exp[-\beta t (\pi n/L)^2].$$

The steady solution is equal to 0.

1.2. **Insulated BC:.**

$$u_t = \beta u_{xx}, \quad u(x,0) = f(x), \quad u_x(0,t) = 0 = u_x(L,t).$$

To solve, expand $f(x)$ in cosine Fourier series; this corresponds to the *even* extension of $f(x)$ to the interval $[-L, L]$, e.g. $f(-x) = f(x)$. The expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{\pi n x}{L}\right), \qquad a_0 = \frac{2}{L}\int_0^L f(x) dx, \qquad a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{\pi n x}{L}\right) dx, n \geq 1.$$

The solution $u(x,t)$ is then given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right) \exp[-\beta t (\pi n/L)^2].$$

The steady solution is equal to $a_0/2$.

2. Inhomogeneous equation

2.1. **Inhomogeneous BC.** For brevity, we shall only consider one kind of inhomogeneous boundary conditions; for solutions in different cases, see section 10 of Professor J.J. Xu's notes.

$$u_t = \beta u_{xx}, \quad u(x,0) = f(x), \quad u(0,t) = u_1,\ u(L,t) = u_2.$$

The *steady solution* $v(x)$ is a linear function satisfying the boundary conditions:

$$v(x) = u_1 + (u_2 - u_1)x/L.$$

The steady solution only depends on the boundary conditions, and does not depend on the initial condition $u(x,0) = f(x)$.

A *general* solution $u(x,t)$ is the sum of the *steady* solution $v(x)$ and a *transient* solution $w(x,t)$, $u(x,t) = v(x) + w(x,t)$.

The transient solution $w(x,t)$ satisfies a *homogeneous* IBVP with *zero* boundary conditions:

$$w_t = \beta w_{xx}, \quad w(x,0) = f(x) - v(x) = f(x) - u_1 - (u_2 - u_1)x/L, \quad w(0,t) = 0 = w(L,t).$$

This equation can be solved as in section 1.1: expand $f(x) - v(x)$ in a sine Fourier series, etc.

2.2. **Inhomogeneous equation (heat source):**

$$u_t = \beta u_{xx} + H(x), \quad u(x,0) = f(x), \quad u(0,t) = u_1,\ u(L,t) = u_2.$$

Solution: We write a general solution $u(x,t)$ as a sum of the steady solution $x(x)$ and a transient solution $w(x,t)$ that decays to 0 as $t \to \infty$. The steady solution $x(x)$ satisfies the following ODE:

$$v''(x) = -H(x)/\beta, v(0) = u_1, v(L) = u_2. \tag{1}$$

A general solution of the 2nd order equation (1) has the form

$$v(x) = -\int_0^x \left(\int_0^z \frac{H(s)}{\beta} ds \right) dz + Ax + B :$$

we integrate the 2nd derivative twice, then add an arbitrary linear function.

The constants A and B can be found from the boundary conditions $v(0) = u_1, v(L) = u_2$. The final formula is

$$v(x) = -\int_0^x \left(\int_0^z \frac{H(s)}{\beta} ds \right) dz + u_1 + \frac{x}{L} \cdot \left[u_2 - u_1 + \int_0^L \left(\int_0^z \frac{H(s)}{\beta} ds \right) dz \right]. \tag{2}$$

If we multiply the coefficient A of x in (2) by L, we get the sum of the temperature difference $u_2 - u_1$ between the endpoints; and the definite integral $\int_0^L \left(\int_0^z \frac{H(s)}{\beta} ds\right) dz$, which is equal to (-1) times the indefinite integral in the expression for $v(x)$, taken with $x = L$. Finally, not that the steady solution $v(x)$ does not depend on the initial condition $u(x,0) = f(x)$.

Once we found the steady solution $v(x)$, we proceed as in section 2.1. Namely, we remark that the transient solution $w(x,t)$ satisfies a *homogeneous* IBVP with *zero* boundary conditions:

$$w_t = \beta w_{xx}, \quad w(x,0) = f(x) - v(x), \quad w(0,t) = 0 = w(L,t).$$

This equation can be solved as in section 1.1: expand $f(x) - v(x)$ in a sine Fourier series, etc.

3. Examples

We next consider several examples of solving inhomogeneous IBVP for the heat equation on the interval:

3.1. **Example 1.** Solve

$$u_t = u_{xx} + e^{-x}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = \sin(2x).$$

Solution: We explain how to find the steady solution $v(x)$, the rest is left to the reader. In this example, $\beta = 1, L = \pi$, and $u_1 = u_2 = 0$. The steady solution $v(x)$ satisfies

$$v''(x) = -e^{-x}, v(0) = v(\pi) = 0.$$

We first compute the indefinite second integral;

$$\int_0^x \left(\int_0^z e^{-s} ds\right) dz = -\int_0^x (1 - e^{-z}) dz = 1 - x - e^{-x}.$$

The steady solution $v(x)$ is then given by

$$v(x) = 1 - x - e^{-x} + 0 + \frac{x}{\pi}(\pi + e^{-\pi} - 1) = \frac{(e^{-\pi} - 1)x}{\pi} - e^{-x} + 1.$$

The transient solution satisfies

$$w_t = w_{xx}, \quad w(x,0) = \sin(2x) - \frac{(e^{-\pi} - 1)x}{\pi} + e^{-x} - 1, \quad w(0,t) = 0 = w(\pi,t).$$

It is solved as in section 1.1.

3.2. **Example 2.** Solve

$$u_t = 2u_{xx} + 4x, \quad u(0,t) = 2, u(\pi,t) = 2 - \pi^3/3, \quad u(x,0) = \sin x.$$

Solution: In this example, $\beta = 2, L = \pi, u_1 = 2, u_2 = 2 - \pi^3/3$. The steady solution $v(x)$ satisfies

$$v''(x) = -4x/2 = -2x, v(0) = 2, v(\pi) = 2 - \pi^3/3.$$

We first compute the indefinite second integral;

$$\int_0^x \left(\int_0^z (-2s) ds \right) dz = \int_0^x (-z^2) dz = -x^3/3.$$

The steady solution $v(x)$ is then given by

$$v(x) = -(x^{/}3) + 2 + \frac{x}{\pi} \left[(2 - \pi^3/3) - 2 + \pi^3/3 \right] = 2 - x^3/3.$$

The transient solution satisfies

$$w_t = 2w_{xx}, \quad w(x,0) = \sin x + x^3/3 - 2, \quad w(0,t) = 0 = w(\pi,t).$$

It is solved as in section 1.1.

3.3. **Example 3.** Solve

$$u_t = u_{xx} + \sin(3x), \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = x.$$

Solution: In this example, $\beta = 1, L = \pi, u_1 = u_2 = 0$. The steady solution $v(x)$ satisfies

$$v''(x) = -\sin(3x), v(0) = v(\pi) = 0.$$

We first compute the indefinite second integral;

$$\int_0^x \left(\int_0^z -\sin(3x) ds \right) dz = \frac{1}{3} \int_0^x (\cos(3z) - 1) dz = \frac{\sin(3x)}{9} - \frac{x}{3}.$$

The steady solution $v(x)$ is then given by

$$v(x) = \frac{\sin(3x)}{9} - \frac{x}{3} + 0 + \frac{x}{\pi} \left(-\frac{\sin(3\pi)}{9} + \frac{\pi}{3} \right) = \frac{\sin(3x)}{9}.$$

The transient solution satisfies

$$w_t = w_{xx}, \quad w(x,0) = x - \sin(3x)/9, \quad w(0,t) = 0 = w(\pi,t).$$

It is solved as in section 1.1.