Math 264: Advanced Calculus **Winter 2008**

Midterm **Thursday, February 21**

Please, solve all 4 problems. Each problem is worth 10 points.

Problem 1. Evaluate the double integral:

$$\iint_S (x+y)\mathrm{d}A,$$

where S is the region in the first quadrant lying inside the disk $x^2+y^2 \le 4$ and under the line $y=\sqrt{3}x$.

Solution: we use polar coordinates $x = r\cos\theta, y = r\sin\theta$. The limits of integration become $0 \le r \le 2, 0 \le \theta \le \pi/3$. The integral is

$$\int_{r=0}^{2} \int \theta = 0^{\pi/3} r(\sin\theta + \cos\theta) r dr d\theta = (r^3/3)_0^2 \cdot (\sin\theta - \cos\theta)_0^{\pi/3} = \frac{4(\sqrt{3}+1)}{3}.$$

Problem 2. Find the volume of the region lying inside the paraboloid $z = 1 - x^2 - y^2$, between the planes $z = 0$ and $z = 1$; and outside the cylinder $x^2 - x + y^2 = 0$. You may use the following identities:

$$\int_0^{\pi/2} \cos^2\theta d\theta = \pi/4; \int_0^{\pi/2} \cos^4\theta d\theta = 3\pi/16.$$

Solution: The equation of the cylinder is $(x-1/2)^2 + y^2 = 1/4$, a circle of radius $1/2$ centered at $(1/2, 0)$ in the x, y-plane; let D_1 denote its interior, the disk $(x-1/2)^2 + y^2 \le 1/4$. Let D denote the disk $x^2 + y^2 \le 1$, and let $\Omega = D \setminus D_1$. In the polar coordinates, Ω can be described as the set of all (r, θ) satisfying $\cos\theta \le r \le 1$ for $-\pi/2 \le \theta \le \pi/2$, and $0, \le r \le 1$ for $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$.

Let R be the region in this problem. we compute its volume in cylindrical coordinates. By symmetry, it suffices to consider $\theta \in [0, \pi]$. The z coordinate satisfies $0 \le z \le 1 - x^2 - y^2 = 1 - r^2$. Accordingly, the volume of R is equal to

$$\int_{\theta=\pi/2}^{\pi} \int_{r=0}^{1} \int_{z=0}^{1-r^2} dz r dr d\theta + \int_{\theta=0}^{\pi/2} \int_{r=\cos\theta}^{1} \int_{z=0}^{1-r^2} dz r dr d\theta := I_1 + I_2.$$

The first integral I_1 is equal to

$$\frac{\pi}{2} \int_{r=0}^{1} (r - r^3) dr = \pi/8.$$

The second integral I_2 is equal to

$$\int_{\theta=0}^{\pi/2} \int_{r=\cos\theta}^{1} (r - r^3) dr d\theta = \int_{\theta=0}^{\pi/2} \left(\frac{1}{4} - \frac{\cos^2\theta}{2} + \frac{\cos^4\theta}{4} \right) d\theta = \frac{3\pi}{64},$$

where we have used the integral identities. The volume is equal to

$$2(I_1 + I_2) = \pi(1/4 + 3/32) = 11\pi/32.$$

Problem 3. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

Solution: Since the problem is symmetric in x and y, we may assume that the cylinder has equation $x^2 + y^2 = 2ax$ or $(x-a)^2 + y^2 = a^2$, which is the circle of radius a centered at $(a, 0)$ in the xy-plane. Denote by D the disk $(x-a)^2 + y^2 \leq a^2$; the part of the sphere lying inside the cylinder projects (two-to-one) onto D. We shall compute the area of the portion S of the sphere lying above the first quadrant in the xy-plane; the total area will be 4 times as large. That part of S project onto the region D_1 in the xy-plane, equal to the intersection of D with the first quadrant. We find that $z = f(x, y) = \sqrt{4a^2 - x^2 - y^2}$.

By the formula proved in class, the area element on S is given by

$$\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy = \sqrt{1 + \frac{x^2 + y^2}{4a^2 - x^2 - y^2}} dxdy = \frac{2adxdy}{\sqrt{4a^2 - x^2 - y^2}},$$

and so the area is equal to

$$A = 8a \int \int_{D_1} \frac{dxdy}{\sqrt{4a^2 - x^2 - y^2}}.$$

We switch to polar coordinates. D_1 becomes the set $\{(r, \theta) : 0 \leq \theta \leq \pi/2, 0 \leq r \leq 2a \cos\theta\}$. The integral becomes

$$8a \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} \frac{rdrd\theta}{\sqrt{4a^2 - r^2}} = 8a \int_{\theta=0}^{\pi/2} \left[-\sqrt{4a^2 - r^2}\right]_{r=0}^{2a\cos\theta}.$$

We get $16a^2 \int_0^{\pi/2} (1 - \sin\theta) d\theta = (8\pi - 16)a^2$.

Problem 4. Consider the vector field

$$\vec{\mathbf{F}}(x, y, z) = (e^x \sin(2y) + y^2 z^3, 2e^x \cos(2y) + 2xyz^3, 3xy^2 z^2)$$

i) Show that $\vec{F}$ is conservative by finding a potential for it.

ii) Evaluate

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}},$$

where $\mathcal{C}$ is the straight line from $(0, 0, 0)$ to $(\pi/4, \pi/4, 1)$.

Solution: We first check the necessary conditions: $\partial F_1/\partial y = 2e^x \cos(2y) + 2yz^3 = \partial F_2/\partial x$; also $\partial F_1/\partial z = 3y^2z^2 = \partial F_3/\partial x$; and finally $\partial F_2/\partial z = 6xyz^2 = \partial F_3/\partial x$, so the necessary conditions hold. They are also sufficient, since $\vec{\mathbf{F}}$ is defined on the whole $\mathbf{R}^3$ that is simply-connected., so the field is conservative.

To find the potential, we want to solve

$$\begin{aligned} \partial\phi/\partial x &= e^x \sin(2y) + y^2 z^3, \\ \partial\phi/\partial y &= 2e^x \cos(2y) + 2xyz^3, \\ \partial\phi/\partial z &= 3xy^2 z^2. \end{aligned} \tag{1}$$

Integrating the last equation with repsect to z, we find that

$$\phi(x, y, z) = xy^2z^3 + f(x, y).$$

Substituting into the 2nd equation we, find that

$$\partial f/\partial y = 2e^x \cos(2y);$$

integrating with respect to y, we find that $f(x, y) = e^x \sin(2y) + g(x)$, and $\phi(x, y, z) = xy^2z^3z + e^x \sin(2y) + g(x)$. Substituting into the first equation, we find that $\partial g/\partial x = 0$, so $g(x) = c$ which we may take to be 0, hence

$$\phi(x, y, z) = xy^2z^3 + e^x \sin(2y).$$

For part (ii), we remark that

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi(\pi/4, \pi/4, 1) - \phi(0, 0, 0) = \frac{\pi^3}{64} + e^{\pi/4}.$$