Department of Mathematics and Statistics
McGill University
Math 262 Practice Midterm
Solutions

INSTRUCTIONS

- You have TWO HOURS to complete the exam.
- Please show how your answers are derived. A correct solution without work will NOT receive full mark.
- Please read each question carefully and answer all questions neatly in the place provided.
- Non-programmable calculators are permitted.
- Formula sheets are not permitted.
- PLEASE NOTE: Invigilators are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

1. Calculate the following limits.

   (a) $\lim_{n\to\infty} \dfrac{4^n}{n!}$

   The limit is equal to 0, since e.g. the ratio $a_{n+1}/a_n = 4/(n+1) \to 0$, so the terms decay faster than a geometric progression.

   (b) $\lim_{n\to\infty} \dfrac{\arctan(n)\cdot(n-1)^n}{n^n}$

   First, $\arctan n \to \pi/2$ as $n \to \infty$. Next, the ratio $(1-1/n)^n \to 1/e$ by a result discussed in class. Thus, the limit is $\pi/(2e)$.

   (c) $\lim_{n\to\infty} \dfrac{n^{1/3}\cdot\ln(n^{-2014})}{(n^2+5n+2)^{1/6}\cdot e^{\ln\ln n}}$.

   The fraction is equal to

$$\frac{-(2014)\ln n}{(1+5/n+2/n^2)^{1/6}\cdot\ln n} \;\to\; (-2014).$$

2. (a) Determine if the series $\sum_{n=1}^{\infty} \sqrt{n}\cdot\sin(1/n)$ converges or diverges. Hint: first compute the limit $n\cdot\sin(1/n)$.

   We first remark that $\lim_{n\to\infty} n\cdot\sin(1/n) = \lim_{n\to\infty}\sin(1/n)/(1/n) = 1$, since $\lim_{x\to 0}\sin x/x = 1$. Accordingly, for large n, $\sqrt{n}\cdot\sin(1/n) \approx (1/\sqrt{n})\cdot(n\cdot\sin(1/n))$. Using the limit comparison test, we find that the series converges or diverges together with $\sum_n 1/\sqrt{n}$. Since the latter series diverges, so does the original series.

   (b) Determine if the series $\sum_{k=3}^{\infty} \dfrac{1}{k(\ln k)^{1/2}}$ converges or diverges.

   We use the integral test. The series converges or diverges together with the integral

$$\int_3^\infty \frac{dx}{x(\ln x)^{1/2}}.$$

   Changing variables $u = \ln x$, the integral becomes $\int_{\ln 3}^\infty du/u^{1/2} = \infty$ since the power of u is equal to $-1/2 > -1$; thus the series diverges.

3. (a) Use the integral test to estimate the difference between the partial sum S_{12} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, and the sum of the series.

The remainder $|R_{12}| = |S - S_{12}|$ is less than or equal to

$$\int_{12}^{\infty} \frac{dx}{x^4} = [-1/(3x^3)]_{12}^{\infty} = 1/(3 \cdot 12^3).$$

(b) Use the Taylor series of $\cos x$ to estimate the difference between $\cos(\pi/5)$ and the number $1 - \pi^2/(2 \cdot 25) + \pi^4/(24 \cdot 5^4)$.

All the derivatives of $f(x) = \cos x$ satisfy $|f^{(k)}(x)| \leq M := 1$. The remainder satisfies

$$|R_n(x)| \leq \frac{Mx^{n+1}}{(n+1)!}.$$

Here $x = \pi/5$ and $n = 4$, so $|R_4| \leq \pi^5/(5^5 \cdot 5!)$. Actually, the 5-th term in the Taylor series is equal to 0, so we also have $|R_5| \leq \pi^6/(5^6 \cdot 6!)$.

4. (a) Find all the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x^2-1)^n}{2^n}$ converges.

We use the root test: $|a_n|^{1/n} = |(x^2-1)/2|$, so the series will converge if $|(x^2-1)/2| < 1$ (case 1); and diverge if $|(x^2-1)/2| > 1$ (case 2). The case $|x^2-1| = 2$ is borderline.

The case 1 is equivalent to $x^2 - 1 \in]-2, 2[$ or, equivalently, $x^2 \in [0, 3[$. This corresponds to $x \in]-\sqrt{3}, +\sqrt{3}[$; for those x the series converges by the root test.

The case 2 is equivalent to $|x^2 - 1| > 2$, or $x^2 > 3$, equivalently $|x| > \sqrt{3}$; for those x the series diverges by the root test.

The borderline case $|x^2-1| = 2$ is equivalent to $x = \pm\sqrt{3}$. In that case, $(x^2-1)/2 = 1$, and the series diverges.

Answer: the series converges for $|x| < \sqrt{3}$, and diverges for $|x| \geq \sqrt{3}$.

(b) Find the sum of the series $\sum_{n=0}^{\infty} x^{n+2}/n!$. Hint: recall the Taylor series of e^x.

We have $e^x = \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2! + x^3/3! + \ldots$ It follows that

$$x^2 e^x = \sum_{n=0}^{\infty} x^{n+2}/n! = x^2 + x^3 + x^4/2! + x^5/3! + \ldots$$

5. (a) Find the Taylor series of the function $f(x) = \int_0^x (y^2 \cos(y))\, dy$ near the point $x = 0$.

The Taylor series of the integrand is given by

$$y^2 \cdot (1 + \sum_{n=1}^{\infty} (-1)^n \cdot y^{2n}/(2n)!) = y^2 + \sum_{n=1}^{\infty} (-1)^n \cdot y^{2n+2}/(2n)!.$$

We next integrate term by term: $\int_0^x y^k dy = [y^{k+1}/(k+1)]_0^x = x^{k+1}/(k+1)$. Substituting into the previous power series, we get

$$f(x) = x^3/3 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n+3}}{(2n+3)\cdot(2n)!}.$$

6. (a) Consider the space curve

$$\mathbf{r}(t) = \langle e^t, \sqrt{2}t, e^{-t}\rangle$$

where $0 \le t \le 2$. Find $\mathbf{T}$ and the curvature at the point $(1, 0, 1)$. Find the length of the curve.

The tangent vector $\mathbf{r}'(t) = \langle e^t, \sqrt{2}, -e^{-t}\rangle$. Its norm is equal to $||\mathbf{r}'(t)|| = \sqrt{e^{2t} + 2 + e^{-2t}} = (e^t + e^{-t})$. The length of the curve is equal to

$$\int_0^2 (e^t + e^{-t})\, dt \;=\; [e^t - e^{-t}]_0^2 = e^2 - e^{-2}.$$

Also

$$\mathbf{T(t)} = \frac{\mathbf{1}}{\mathbf{e^t - e^{-t}}} \cdot \langle \mathbf{e^t}, \sqrt{\mathbf{2}}, -\mathbf{e^{-t}}\rangle.$$

The point $(1, 0, 1)$ corresponds to $t = 0$. We have $\mathbf{T}(0) = \langle 1/2, 1/\sqrt{2}, -1/2\rangle$. We also have $\mathbf{r}'(0) = \langle 1, \sqrt{2}, -1\rangle$ and $||\mathbf{r}'(0)|| = 2$.

Next, $\mathbf{r}''(t) = \langle e^t, 0, e^{-t}\rangle$, $\mathbf{r}''(0) = \langle 1, 0, 1\rangle$. The curvature at $t = 0$ is given by

$$\kappa(0) = \frac{||\mathbf{r}'(0) \times \mathbf{r}''(0)||}{||\mathbf{r}'(0)||^3} = \frac{1}{8}||\langle\sqrt{2}, -2, -\sqrt{2}\rangle|| = \frac{\sqrt{8}}{8} = \frac{1}{2\sqrt{2}}.$$