McGill University — Math 262: Intermediate Calculus, Fall 2014

WRITTEN ASSIGNMENT 2 — Solutions

Problem 1. **(4 points)** Consider the space curve $\mathbf{r}(t) = (3t^2, 6t, 3\ln t)$, where $1 \le t \le 3$.

a) Find $\mathbf{r}'(t), \mathbf{r}''(t)$; compute the arc-length of $\mathbf{r}(t)$.

b) Find $\mathbf{T}$ and the curvature κ at $t = 1$.

Solution:

a) We'll start with finding $\mathbf{r}'(t)$. This can be done by deriving each component of $\mathbf{r}(t)$ in terms of t :

$$\mathbf{r}'(t) = (6t, 6, \frac{3}{t}).$$

Now, to find the second derivative, $\mathbf{r}''(t)$, we take the derivative of each component of $\mathbf{r}'(t)$ in terms of t:

$$\mathbf{r}''(t) = (6, 0, \frac{-3}{t^2}).$$

Recall the arc-length equation given by,

$$s(t) = \int^t \mid r'(u) \mid du.$$

In this case, this is calculated as follows,

$$\begin{aligned} s(t) &= \int_1^3 \sqrt{(6t)^2 + (6)^2 + (\frac{3}{t})^2 dt} \\ &= \int_1^3 \sqrt{36t^2 + 36 + \frac{9}{t^2} dt} \\ &= \int_1^3 \frac{1}{t}\sqrt{36t^4 + 36t^2 + 9} dt \\ &= \int_1^3 \frac{1}{t}\sqrt{(6t^2+3)^2} dt \\ &= \int_1^3 \frac{1}{t}(6t^2+3) dt \\ &= \int_1^3 6t + \frac{3}{t} dt \end{aligned}$$

$$= [3t^2 + 3\ln t] \,|_1^3$$
$$= 24 + 3\ln 3.$$

b) Recall that the unit tangent vector $\mathbf{T}$ is found through the equation,

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\ \mathbf{r}'(t)\ |}.$$

From part a), we have $\mathbf{r}'(t)$ so we just need to find the magnitude,

$$|\ \mathbf{r}'(t)\ | = \frac{1}{t}(6t^2 + 3).$$

Now, putting $\mathbf{r}'(t)$ and $|\ \mathbf{r}'(t)\ |$ back into the equation for the unit tangent, we have,

$$\mathbf{T} = \frac{(6t, 6, \frac{3}{t})}{\frac{1}{t}(6t^2 + 3)} = \frac{(6t^2, 6t, 3)}{6t^2 + 3}.$$

When $t = 1$,

$$\mathbf{T}(1) = \frac{(6, 6, 3)}{9} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}).$$

The curvature is given by the equation,

$$\begin{aligned}
\kappa(t) &= \frac{|\ \mathbf{r}'(t) \times \mathbf{r}''(t)\ |}{|\ \mathbf{r}'(t)\ |^3} \\
&= \frac{\left|\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6t & 6 & \frac{3}{t} \\ 6 & 0 & -\frac{3}{t^2} \end{vmatrix}\ \right|}{(\frac{1}{t}(6t^2 + 3))^3} \\
&= \frac{|\ (-\frac{18}{t^2}, \frac{36}{t}, -36)\ |}{\frac{1}{t^3}(6t^2 + 3)^3} \\
&= \frac{\sqrt{(\frac{18}{t^2})^2 + (\frac{36}{t})^2 + 36^2}}{\frac{1}{t^3}(6t^2 + 3)^3} \\
\kappa(1) &= \frac{\sqrt{(18)^2 + (36)^2 + 36^2}}{(9)^3} = \frac{2}{27}.
\end{aligned}$$

Problem 2. (4 points) Find $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for the following curves:

a) $\mathbf{r}(t) = (t^2, 2t^3/3, t)$ at $t = 1$;

b) $\mathbf{r}(t) = (\cos t, \sin t, \ln \cos t)$ at $t = 0$.

Solution:

a) To calculate the unit tangent, we need to find $\mathbf{r}(t)$ and $\mid \mathbf{r}(t) \mid$:

$$\mathbf{r}'(t) = (2t, 2t^2, 1)$$

and

$$\mid \mathbf{r}'(t) \mid = \sqrt{(2t)^2 + (2t^2)^2 + 1^2} = \sqrt{4t^4 + 4t^2 + 1} = \sqrt{(2t^2+1)^2} = 2t^2 + 1.$$

The unit tangent is therefore given by,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\mid \mathbf{r}(t) \mid} = \frac{1}{2t^2+1}(2t, 2t^2, 1).$$

Evaluating the above at $t = 1$ gives,

$$\mathbf{T}(1) = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}).$$

To find the unit normal, we need $\mathbf{T}'$ and $\mid \mathbf{T}' \mid$:

$$\begin{aligned}\mathbf{T}'(t) &= (\frac{2}{2t^2+1} - \frac{2t(4t)}{(2t^2+1)^2}, \frac{4t}{2t^2+1} - \frac{2t^2(4t)}{(2t^2+1)^2}, -\frac{4t}{(2t^2+1)^2}) \\ &= (\frac{2}{2t^2+1} - \frac{8t^2}{(2t^2+1)^2}, \frac{4t}{2t^2+1} - \frac{8t^3}{(2t^2+1)^2}, -\frac{4t}{(2t^2+1)^2}).\end{aligned}$$

Now,

$$\mathbf{T}'(1) = (\frac{2}{3} - \frac{8}{9}, \frac{4}{3} - \frac{8}{9}, -\frac{4}{9}) = (-\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}),$$

and

$$\mid \mathbf{T}'(1) \mid = \sqrt{(\frac{2}{9})^2 + (\frac{4}{9})^2 + (\frac{4}{9})^2} = \frac{2}{3}.$$

Substituting the above into the equation for $\mathbf{N}$ gives,

$$\mathbf{N} = \frac{(-\frac{2}{9}, \frac{4}{9}, -\frac{4}{9})}{\frac{2}{3}} = (-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}).$$

The unit binormal is calculated by taking the cross product of the tangent and the normal vectors,

$$\begin{aligned}
\mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} \\
&= (\frac{-4}{9} - \frac{2}{9}, \frac{4}{9} - \frac{1}{9}, \frac{4}{9} + \frac{2}{9}) \\
&= (\frac{-2}{3}, \frac{1}{3}, \frac{2}{3})
\end{aligned}$$

b) To calculate the unit tangent, we need to find $\mathbf{r}(t)$ and $\mid \mathbf{r}(t) \mid$:

$$\begin{aligned}
\mathbf{r}'(t) &= (-\sin t, \cos t, \frac{1}{\cos t}(-\sin t)) \\
&= (-\sin t, \cos t, -\tan t)
\end{aligned}$$

and

$$\begin{aligned}
\mid \mathbf{r}'(t) \mid &= \sqrt{(\sin t)^2 + (\cos t)^2 + (-\tan t)^2} \\
&= \sqrt{1 + \tan^2 t} \\
&= \sqrt{\sec^2 t} \\
&= \sec t.
\end{aligned}$$

The unit tangent is therefore given by,

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{\mid \mathbf{r}(t) \mid} \\
&= (\frac{-\sin t}{\sec t}, \frac{\cos t}{\sec t}, \frac{-\tan t}{\sec t}) \\
&= (-\sin t \cos t, \cos^2 t, -\sin t).
\end{aligned}$$

Evaluating the above at $t = 0$ gives,

$$\mathbf{T}(0) = (-\sin 0 \cos 0, \cos^2 0, -\sin 0) = (0, 1, 0).$$

The unit normal vector requires $\mathbf{T}'(t)$ which is given by,

$$\mathbf{T}'(t) = (-\cos^2 t + \sin^2 t, -2\sin t\cos t, -\cos t),$$

evaluating this at $t = 0$ gives

$$\mathbf{T}'(0) = (-1, 0, -1)$$

with magnitude

$$\mid \mathbf{T}'(0) \mid = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

The unit normal is therefore given by,

$$\mathbf{N}(0) = \frac{(-1, 0, -1)}{\sqrt{2}} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}).$$

The binormal can be calculated in the same way as previously done and is given by

$$\begin{aligned}\mathbf{B}(0) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} \\ &= (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\end{aligned}$$

Problem 3. (4 points) Verify the following identities:

a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$;

b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$.

You can use any properties of the determinant that you know.
Proof:

a) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ with respect to the standard basis.

From this, we have

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k},$$

then we have

$$
\begin{aligned}
\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_2w_3 - v_3w_2 & -v_1w_3 + v_3w_1 & v_1w_2 - v_2w_1 \end{vmatrix} \\
&= \big(u_2(v_1w_2 - v_2w_1) - u_3(-v_1w_3 + v_3w_1)\big)\mathbf{i} \\
&\quad - \big(u_1(v_1w_2 - v_2w_1) - u_3(v_2w_3 - v_3w_2)\big)\mathbf{j} \\
&\quad + \big(u_1(-v_1w_3 + v_3w_1) - u_2(v_2w_3 - v_3w_2)\big)\mathbf{k} \\
&= \big((\mathbf{u}\cdot\mathbf{w})v_1 - (\mathbf{u}\cdot\mathbf{v})w_1\big)\mathbf{i} \\
&\quad + \big((\mathbf{u}\cdot\mathbf{w})v_2 - (\mathbf{u}\cdot\mathbf{v})w_2\big)\mathbf{j} \\
&\quad + \big((\mathbf{u}\cdot\mathbf{w})v_3 - (\mathbf{u}\cdot\mathbf{v})w_3\big)\mathbf{k} \\
&= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}.
\end{aligned}
$$

b) Note that

$$
\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})\cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
\end{aligned}
$$

With the properties of determiant, say, if two rows of a determinant interchanged, then the determinant changes signs, the identity $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$ follows.

Problem 4. (4 points)

a) Find the distance between the lines $x + 2y = 3, y + 2z = 3$ and $x + y + z = 6, x - 2z = -5$.

b) Show that the line $x - 2 = (y + 3)/2 = (z - 1)/4$ is parallel to the plane $2y - z = 1$. What is the distance between the line and the plane?

Solution:

a) The two plans $x + 2y = 3, y + 2z = 3$ have normals

$$\mathbf{n}_1 = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{n}_2 = \mathbf{j} + 2\mathbf{k}.$$

Thus a direction vector of the line l_1 of their intersection is

$$\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = (\mathbf{i} + 2\mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Let $z = 0$, then $P_1 = (-3, 3, 0) \in l_1$;
Similarly, the two plans $x + y + z = 6, x - 2z = -5$ have normals

$$\mathbf{n}_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{n}_4 = \mathbf{i} - 2\mathbf{k}.$$

Thus a direction vector of the line l_2 of their intersection is

$$\mathbf{v}_2 = \mathbf{n}_3 \times \mathbf{n}_4 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k},$$

Let $z = 0$, then $P_2 = (-5, 11, 0) \in l_2$.
Therefore

$$\mathbf{v}_1 \times \mathbf{v}_2 = (4\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = -\mathbf{i} + 2\mathbf{j} + 8\mathbf{k},$$

then the distance between the line l_1 and l_2 is given by

$$\begin{aligned} s &= \frac{|\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{|\mathbf{v}_1 \times \mathbf{v}_2|} \\ &= \frac{1}{\sqrt{1 + 2^2 + 8^2}} |(-2\mathbf{i} + 8\mathbf{j}) \cdot (-\mathbf{i} + 2\mathbf{j} + 8\mathbf{k})| \\ &= \frac{18}{\sqrt{69}} \\ &= \frac{6\sqrt{69}}{23}. \end{aligned}$$

b) Note that the line l is the intersection of plans $2(x - 2) = y + 3$ and $2(y + 3) = z - 1$. Their respective normals is

$$\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j}, \quad \mathbf{n}_2 = 2\mathbf{j} - \mathbf{k},$$

Thus a direction vector of the line l is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (2\mathbf{i} - \mathbf{j}) \times (2\mathbf{j} - \mathbf{k}) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

On the other hand, the plane H defined by $2y - z = 1$ has normal

$$\mathbf{n} = 2\mathbf{j} - \mathbf{k}.$$

Since

$$\mathbf{v} \cdot \mathbf{n} = (\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 0,$$

therefore, the line l is parallel to the plane H.

Moreover, since the line l is parallel to the plane H, the distance between l and H equals the distance from any point $P \in l$ to H. We may take $P = (2, -3, 1) \in l$, then

$$\text{distance between } l \text{ and } H = \frac{|2 \cdot (-3) - 1 - 1|}{\sqrt{0^2 + 2^2 + 1^2}} = \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}.$$

Problem 5. (4 points)

a) Express the length of the curve $\mathbf{r} = (at^2, bt, c \cdot \ln t), 1 \le t \le T$ as a definite integral. Evaluate the integral if $b^2 = 4ac$.

b) Find the arc length parametrization of the curve $\mathbf{r} = (3t\cos t, 3t\sin t, 2\sqrt{2}t^{3/2})$.

Solution:

a) We compute

$$\mathbf{r}'(t) = (2at, b, \frac{c}{t}),$$

then

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}} = \sqrt{(2at + \frac{c}{t})^2 + (b^2 - 4ac)},$$

Therefore

$$\text{arc length} = \int_1^T \sqrt{(2at + \frac{c}{t})^2 + (b^2 - 4ac)}dt.$$

In particular, if $b^2 = 4ac$ (Since $ac \ge 0$, let us assume $a, c \ge 0$), then

$$\text{arc length} = \int_1^T (2at + \frac{c}{t})dt = (at^2 + c\ln t)\Big|_1^T = a(T^2 - 1) + c\ln T.$$

b) Note that the curve is defined on $[0, +\infty)$, then

$$\frac{d\mathbf{r}}{dt} = (3(\cos t - t\sin t), 3(\sin t + t\cos t), 3\sqrt{2t})$$

and thus

$$\begin{aligned}\frac{ds}{dt} &= \left|\frac{d\mathbf{r}}{dt}\right| \\ &= 3\sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t} \\ &= 3\sqrt{1 + t^2 + 2t} \\ &= 3(t+1) > 0,\end{aligned}$$

thus $s(t)$ is a strictly increasing function of t, and $t = t(s)$ can be parametrized in terms of arc length s either. Moreover, $s(0) = 0$, thus

$$s = \frac{3}{2}t^2 + 3t.$$

Solve t in terms of s, then

$$t = \sqrt{1 + \frac{2}{3}s} - 1.$$

From this, we obtain the arc length parametrization:

$$\mathbf{r}(s) = (3t\cos t, 3t\sin t, 2\sqrt{2}t^{3/2}),$$

where $t = \sqrt{1 + \frac{2}{3}s} - 1$, $s \geq 0$.

Problem 6. (4 points)

Find $\mathbf{T}, \mathbf{N}, \mathbf{B}$, curvature and torsion at a general point on the curve $\mathbf{r} = (e^t \cos t, e^t \sin t, e^t)$.

Solution:

(I) Write

$$\mathbf{r}(t) = e^t(\cos t, \sin t, 1),$$

we first compute the velocity,

$$\frac{ds}{dt} = \mathbf{r}'(t) = e^t(\cos t, \sin t, 1) + e^t(-\sin t, \cos t, 0) = e^t(\cos t - \sin t, \sin t + \cos t, 1)$$

then we obtain

$$|\mathbf{r}'(t)| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = \sqrt{3}e^t.$$

The Frenet frame is

$$\begin{aligned}
\widehat{\mathbf{T}}(t) &= \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) \\
&= \frac{1}{\sqrt{3}}(\cos t - \sin t, \sin t + \cos t, 1) \\
\widehat{\mathbf{N}}(t) &= \frac{T'(t)}{|T'(t)|} \\
&= \frac{1}{\sqrt{2}}(-\sin t - \cos t, \cos t - \sin t, 0) \\
\widehat{\mathbf{B}}(t) &= \widehat{\mathbf{T}}(t) \times \widehat{\mathbf{N}}(t)
\end{aligned}$$

$$= \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -\sin t - \cos t & \cos t - \sin t & 0 \end{vmatrix}$$
$$= \frac{1}{\sqrt{6}}(\sin t - \cos t, -\sin t - \cos t, 2)$$

The curvature is

$$\kappa(t) = |\frac{dt}{ds}\frac{d}{dt}\widehat{\mathbf{T}}| = \frac{|\widehat{\mathbf{T}}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\frac{1}{\sqrt{3}}(-\sin t - \cos t, \cos t - \sin t, 0)|}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3}e^{-t},$$

Next we compute the torsion,

$$\frac{d}{dt}\widehat{\mathbf{B}} = \frac{1}{\sqrt{6}}(\sin t + \cos t, \sin t - \cos t, 2).$$

With the definition of torsion

$$\frac{d}{dt}\widehat{\mathbf{B}} = \frac{ds}{dt}\frac{d}{ds}\widehat{\mathbf{B}} = -|\mathbf{r}'(t)|\tau\widehat{\mathbf{N}},$$

therefore

$$\tau(t) = \frac{1}{3}e^{-t}.$$

(II) Alternatively, we apply the general formula with respect to general parametrization. Write

$$\mathbf{r}(t) = e^t(\cos t, \sin t, 1),$$

we first compute

$$\begin{aligned} \mathbf{r}'(t) &= e^t(\cos t, \sin t, 1) + e^t(-\sin t, \cos t, 0) \\ &= e^t(\cos t - \sin t, \sin t + \cos t, 1) \\ \mathbf{r}''(t) &= e^t(\cos t - \sin t, \sin t + \cos t, 1) + e^t(-\sin t - \cos t, \cos t - \sin t, 0) \\ &= e^t(-2\sin t, 2\cos t, 1) \\ \mathbf{r}'''(t) &= e^t(-2\sin t, 2\cos t, 1) + e^t(-2\cos t, -2\sin t, 0) \\ &= e^t(-2(\sin t + \cos t), -2(\sin t - \cos t), 1) \end{aligned}$$

From these, we obtain

$$|\mathbf{r}'(t)| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = \sqrt{3}e^t,$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = e^t e^t \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 1 \end{vmatrix}$$
$$= e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2).$$

The Frenet frame can be determined as

$$\begin{aligned}
\widehat{\mathbf{T}}(t) &= \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) \\
&= \frac{1}{\sqrt{3}}(\cos t - \sin t, \sin t + \cos t, 1) \\
\widehat{\mathbf{B}}(t) &= \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|} \\
&= \frac{1}{\sqrt{6}}(\sin t - \cos t, -\sin t - \cos t, 2) \\
\widehat{\mathbf{N}}(t) &= \widehat{\mathbf{B}}(t) \times \widehat{\mathbf{T}}(t) \\
&= \frac{1}{\sqrt{6}}\frac{1}{\sqrt{3}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t - \cos t & -\sin t - \cos t & 2 \\ \cos t - \sin t & \sin t + \cos t & 1 \end{vmatrix} \\
&= \frac{1}{\sqrt{2}}(-\sin t - \cos t, \cos t - \sin t, 0).
\end{aligned}$$

The curvature is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{6}e^{2t}}{(\sqrt{3}e^t)^3} = \frac{\sqrt{2}}{3}e^{-t},$$

and the torsion is

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} = \frac{2e^{3t}}{(\sqrt{6}e^{2t})^2} = \frac{1}{3}e^{-t}.$$