McGill University — Math 262: Intermediate Calculus, Fall 2014

WRITTEN ASSIGNMENT 1 — Solutions

Problem 1.

Find the limit of the following sequences.

(a)

$$a_n = \left(\frac{n-3}{n}\right)^n$$

(b)

$$a_n = (n^2+2n+3) \cdot \sin\left(\frac{1}{4n^2+5n+3}\right).$$

Solution, Part a). We stated in class that $\lim_{n\to\infty}(1+a/n)^n = e^a$. Here we have $(n-3)/n = 1-3/n$, so substituting $a=-3$ we find that the answer is $e^{-3} = 1/e^3$.

Solution, Part b). It is easy to show using Taylor expansion of sin that $\lim_{x\to 0} \cdot \sin(x)/x = 0$. Let us re-write the expression as

$$(4n^2+5n+3)\sin\left(\frac{1}{4n^2+5n+3}\right) \cdot \frac{n^2+2n+3}{4n^2+5n+3}.$$

If we let $x(n) = 1/(4n^2+5n+3)$, we find that $x(n) \to 0$ as $n \to \infty$, and it follows from our first remark that $(4n^2+5n+3)\sin\left(\frac{1}{4n^2+5n+3}\right) \to 1$ as $n \to \infty$. Next, the fraction

$$\frac{n^2+2n+3}{4n^2+5n+3} \to \frac{1}{4}$$

as $n \to \infty$. Accordingly, the limit is equal to $1/4$.

Problem 2.

Determine whether the following series converges or diverges by using any appropriate test.

(a)

$$\sum_{n=1}^{\infty} \frac{n^n}{\pi^n \cdot n!}$$

(b)

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

Solution, Part a). We use the ratio test. We find that

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{\pi} \to \frac{e}{\pi}$$

as $n \to \infty$. Since $|e/\pi| < 1$, we find that the series converges by the ratio test.
Solution, Part b). We use the root test. We find that

$$a_n^{1/n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$$

as $n \to \infty$. Since $|1/e| < 1$, we find that the series converges by the root test.
Problem 3.

Determine whether the following series converges or diverges.

(a)

$$\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$$

Hint: compare a_n with $1/(2n)$.

(b)

$$\sum_{n=27}^{\infty} \frac{1}{n \cdot \ln n (\ln \ln n)(\ln \ln \ln n)^{1.5}}$$

Solution, Part a). Let $a_n = b_n/(2n)$, so $b_n = 4^n(n!)^2/(2n-1)!$. We claim that $b_{n+1} > b_n$ for all n. Indeed, we find after cancellations that

$$\frac{b_{n+1}}{b_n} = \frac{4(n+1)^2}{(2n+1) \cdot 2n} = \frac{4n^2+8n+4}{4n^2+4n} > 1.$$

It follows that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{b_n}{2n} \geq b_1 \sum_{n=1}^{\infty} \frac{1}{2n} = \infty$$

by the p-series test. So, $\sum_n a_n = \infty$ by the comparison principle.
Solution, Part b). We use the integral test. We have to compute

$$\int_A^{\infty} \frac{dx}{x \cdot \ln x \cdot \ln(\ln x) \cdot (\ln(\ln(\ln x)))^{1.5}}.$$

We let $u = \ln \ln \ln x$. By the chain rule applied repeatedly, we find that $du = dx/(x \cdot \ln x \cdot \ln(\ln x))$. The integral becomes

$$\int_B^{\infty} \frac{du}{u^{1.5}} < \infty$$

by the p-series test. The series thus converges by the integral test.

Problem 4. *Fibonacci numbers* f_n are defined as follows: $f_1 = 1, f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Consider the power series

$$F(x) = \sum_{n=1}^{\infty} f_n x^n.$$

Show that

$$F(x) = \frac{x}{1-x-x^2}.$$

Hint: Multiply $F(x)$ by $(1-x-x^2)$ and use the recursion relations.

Solution. We find that $F(x)(1-x-x^2)$ is equal to

$$\begin{aligned} x + x^2 + 2x^3 &\sum_{n=4}^{\infty} f_n x^n \\ -x^2 - x^3 - &\sum_{n=4}^{\infty} f_{n-1} x^n \\ - x^3 - &\sum_{n=4}^{\infty} f_{n-2} x^n = \end{aligned}$$

$$x + x^2(1-1) + x^3(2-1-1) + \sum_{n=4}^{\infty}(f_n - f_{n-1} - f_{n-2})x^n.$$

Using the recursion relation $f_n = f_{n-1} + f_{n-2}$ we see that the last expression is equal to x. The result follows after dividing by $(1-x-x^2)$.

Problem 5. Let $F(x) = \int_0^x \sin(t^2)dt$.

- Find the Maclaurin series for $F(x)$.
- Approximate $F(0.1)$ with an error smaller than 0.001.

Solution. The Taylor series for $\sin y$ is $y - y^3/3! + y^5/5! + \ldots + (-1)^n y^{2n+1}/(2n+1)! + \ldots$ Substituting $y = t^2$

$$t^2 - t^6/3! + t^{10}/5! + \ldots + (-1)^n t^{4n+2}/(2n+1)! + \ldots$$

Integrating term by term gives

$$[t^3/3 - t^7/(3! \cdot 7) + \ldots + (-1)^n t^{4n+3}/((4n+3)\cdot(2n+1)!)]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)\cdot(2n+1)!}$$

This is an alternating series, and the remainder $R_n = \pm(a_{n+1} - a_{n+2} \pm \ldots)$ satisfies $|R_n| \leq |a_{n+1}|$, so it suffices to find n such that $(.1)^{4n+3}/[(4n+3)\cdot(2n+1)!] < .001$. It is easy to show that for $n = 1$ the inequality is satisfied, so we can choose an approximation consisting of just the first term in the sum, $.1^3/3$. However, even that leading term is less than .001, so we can leave 0 as an approximation as well.

Problem 6.

a) Find

$$\lim_{x\to 0} \frac{\sin(\sin x) - x}{x(\cos(\sin x) - 1)}.$$

b) Find MacLaurin series representation for the function

$$(4 + x^4)^{-1/3}.$$

Solution, Part a). Let $y = sinx = x - x^3/3! + x^5/5! \pm \ldots$. We first compute the numerator. We see that $\sin y = y - y^3/3! + y^5/5! \pm \ldots$ Substituting for y, we find that $-x + \sin(\sin x)$ is equal to

$$-x + x - x^3/6 + x^5/5! \pm \ldots - \frac{1}{6}(x^3 - 3x^5/6 \pm \ldots) + \frac{1}{5!}(x^5 + \ldots) = -x^3/3 \pm \ldots$$

The denominator is equal to $x(-1 + \cos y) = x(-y^2/2 + y^4/4! \pm \ldots)$ Substituting for y, we see that it is equal to

$$x(\frac{-1}{2}(x^2 - 2x^4/6 \pm \ldots + x^4/24 \pm \ldots)) = -x^3/2 \pm \ldots$$

The limit is equal to

$$\lim_{x\to 0} \frac{-x^3/3 \pm \ldots}{-x^3/2 \pm \ldots} = \frac{2}{3}.$$

Solution, Part b). Let $y = x^4/4$. We find that $(4+x^4)^{-1/3} = 4^{-1/3}(1+y)^{-1/3}$. The binomial power series for $(1+y)^{-1/3}$ gives

$$1 + \frac{-y}{3} + \frac{(-1/3)(-1/3-1)y^2}{2!} + \ldots + \frac{(-1/3)(-1/3-1)\ldots(-1/3-n+1)y^n}{n!} + \ldots$$

Multiplying by $4^{-1/3}$ and substituting $y = x^4/4$ gives

$$4^{-1/3}[1 + \frac{-x^4}{3\cdot 4} + \frac{(-1/3)(-1/3-1)x^8}{2!\cdot 4^2} + \ldots + \frac{(-1/3)(-1/3-1)\ldots(-1/3-n+1)x^{4n}}{n!\cdot 4^n} + \ldots]$$