McGill University
Math 325A: Differential Equations

LECTURE 22: EIGENVECTOR METHOD: SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS (II)

(Text: Chap. 5)

1 Introduction

In this lecture we will apply the eigenvector method to the solution of a second order system of the type arising in the solution of a mass-spring system with two masses. The system we will consider consists of two masses with mass m_1, m_2 connected by a spring with spring constant k_2. The first mass is attached to the ceiling of a room by a spring with spring constant k_1 and the second mass is attached to the floor by a spring with spring constant k_3 at a point immediately below the point of attachment to the ceiling. Assume that the system is under tension and in equilibrium. If $x_1(t)$, $x_1(t)$ are the displacements of the two masses from their equilibrium position at time t, the positive direction being upward, then the motion of the system is determined by the system

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= -k_1x_1 - k_2(x_1-x_2) = -(k_1+k_2)x_1 + k_2x_2, \\ m_2\frac{d^2x_2}{dt^2} &= k_2(x_1-x_2) - k_3x_2 = k_2x_1 - (k_2+k_3)x_2. \end{aligned}$$

The system can be written in matrix form $\frac{d^2X}{dt^2} = AX$ where

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_2 \\ k_2/m_1 & -(k_2+k_3)/m_2 \end{bmatrix}.$$

The characteristic polynomial of A is

$$r^2 + \frac{m_2(k_1+k_2)+m_1(k_2+k_3)}{m_1m_2}r + (\frac{(k_1+k_2)(k_2+k_3)}{m_1m_2} - \frac{k_2^2}{m_1m_2}).$$

The discriminant of this polynomial is

$$\begin{aligned} \Delta &= \frac{(m_2(k_1+k_2)+m_1(k_2+k_3))^2 - 4(k_1+k_2)(k_2+k_3)m_1m_2 + 4k_2^2m_1m_2}{m_1^2m_2^2} \\ &= \frac{(m_2(k_1+k_2)-m_1(k_2+k_3))^2 + 4m_1m_2{k_2}^2}{m_1^2m_2^2} > 0. \end{aligned}$$

Hence the eigenvalues of A are real, distinct and negative since the trace of A is negative while the determinant is positive. Let $r_1 > r_2$ be the eigenvalues of A and let

$$P_1 = \begin{bmatrix} 1 \\ s_1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ s_2 \end{bmatrix}$$

be (normalized) eigenvectors with eigenvalues r_1, r_2 respectively. We have

$$s_1 = \frac{m_1 r_1 + k_1 + k_2}{k_2}, \quad s_2 = \frac{m_1 r_2 + k_1 + k_2}{k_2}$$

and, if P is the matrix with columns P_1, P_2, we have

$$P^{-1}AP = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

If we make a change of variables $X = PY$ with $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we have

$$\frac{d^2Y}{dt^2} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

so that our system in the new variables y_1, y_2 is

$$\begin{aligned} \frac{d^2y_1}{dt^2} &= r_1 y_1 \\ \frac{d^2y_2}{dT^2} &= r_2 y_2. \end{aligned}$$

Setting $r_i = -\omega_i^2$ with $\omega_i > 0$, this uncoupled system has the general solution

$$y_1 = A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t), \quad y_2 = A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t).$$

Since $X = PY = y_1 P_1 + y_2 P_2$, we obtain the general solution

$$X = (A_1 \sin(\omega_1 t) + B_1 \cos(\omega_1 t))P_1 + (A_2 \sin(\omega_2 t) + B_2 \cos(\omega_2 t))P_2.$$

The two solutions with $Y(0) = P_i$ are of the form

$$X = (A \sin(\omega_i t) + B \cos(\omega_i t))P_i = \sqrt{A^2 + B^2} \sin(\omega_i t + \theta_i)P_i.$$

These motions are simple harmonic with frequencies $\omega_i/2\pi$ and are called the **fundamental motions** of the system. Since any motion of the system is the sum (superposition) of two such motions any periodic motion of the system must have a period which is an integer multiple of both the fundamental periods $2\pi/\omega_1$, $2\pi/\omega_2$. This happens if and only if ω_1/ω_2 is a rational number. If $X'(0) = 0$, the fundamental motions are of the form

$$X = B_i \cos(\omega_i t)P_i$$

and if $X(0) = 0$ they are of the form

$$X = A_i \sin(\omega_i t)P_i.$$

These four motions are a basis for the solution space of the given system. The motion is completely determined once $X(0)$ and $X'(0)$ are known since

$$X(0) = PY(0) = P \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad X'(0) = PY'(0) = P \begin{bmatrix} \omega_1 A_1 \\ \omega_2 A_2 \end{bmatrix}.$$

As a particular example, consider the case where $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. The system is symmetric and

$$A = \frac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$

a symmetric matrix. The characteristic polynomial is

$$r^2 + 4\frac{k}{m}r + 3\frac{k^2}{m^2} = (r + \frac{k}{m})(r + 3\frac{k}{m}).$$

The eigenvalues are $r_1 = -k/m$, $r_2 = -3k/m$. The fundamental frequencies are $\omega_1 = \sqrt{k/m}$, $\omega_2 = \sqrt{3k/m}$. The normalized eigen vectors are

$$P_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The fundamental motions with $X'(0) = 0$ are

$$X = A\cos(\sqrt{k/m}\; t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad X = A\cos(\sqrt{3k/m}\; t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the ratio of the fundamental frequencies is $\sqrt{3}$, an irrational number, theses are the only two periodic motions of the mass-spring system where the masses are displaced and then let go.

Odds and Ends

If $y = f(x)$ is a solution of the autonomous DE $y^n = f(y, y', \ldots, y^{n-1})$ then so is $y = f(x + a)$ for any real number a. If the DE is linear and homogeneous with fundamental set $y_1, y_2, \ldots, y_n$ then we must have identities of the form

$$y_1(x + a) = c_2y_2 + c_3y_3 + \cdots + c_ny_n.$$

For example, consider the DE $y'' + y = 0$. Here $\sin(x), \cos(x)$ is a fundamental set so we must have an identity of the form

$$\sin(x + a) = A\sin(x) + B\cos(x).$$

Differentiating, we get $\cos(x + a) = A\cos(x) - B\sin(x)$. Setting $x = 0$ in these two equations we find $A = \cos(a)$, $B = \sin(a)$. We obtain in this way the addition formulas for the sine and cosine functions:

$$\sin(x + a) = \sin(x)\cos(a) + \sin(a)\cos(x), \quad \cos(x + a) = \cos(x)\cos(a) - \sin(x)\sin(a).$$

The numerical methods for solving DE's can be extended to systems virtually without change. In this way we can get approximate solutions for higher order DE's. For more details consult the text (Chapter 5).