McGill University
Math 325A: Differential Equations

LECTURE 21: INTRODUCTION TO SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS (I)

(Text: Chap. 5)

1 Introduction

In this and the following lecture we will give an introduction to systems of differential equations. For simplicity, we will limit ourselves to systems of two equations with two unknowns. The techniques introduced can be used to solve systems with more equations and unknowns. As a motivational example, consider the the following problem.

1.1 Mathematical Formulation of a Practical Problem

Two large tanks, each holding 24 liters of brine, are interconnected by two pipes. Fresh water flows into tank A a the rate of 6 L/min, and fluid is drained out tank B at the same rate. Also, 8 L/min of fluid are pumped from tank A to tank B and 2 L/min from tank B to tank A. The solutions in each tank are well stirred sot that they are homogeneous. If, initially, tank A contains 5 in solution and Tank B contains 2 kg, find the mass of salt in the tanks at any time t.

To solve this problem, let $x(t)$ and $y(t)$ be the mass of salt in tanks A and B respectively. The variables x, y satisfy the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{-1}{3}x + \frac{1}{12}y, \\ \frac{dy}{dt} &= \frac{1}{3}x - \frac{1}{3}y. \end{aligned}$$

The first equation gives $y = 12\frac{dx}{dt} + 4x$. Substituting this in the second equation and simplifying, we get

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{dx}{dt} + \frac{1}{12}x = 0.$$

The general solution of this DE is

$$x = c_1e^{-t/2} + c_2e^{-t/6}.$$

This gives $y = 12\frac{dx}{dt} + 4x = -2c_1e^{-t/2} + 2c_2e^{-t/6}$. Thus the general solution of the system is

$$\begin{aligned} x &= c_1e^{-t/2} + c_2e^{-t/6}, \\ y &= -2c_1e^{-t/2} + 2c_2e^{-t/6}. \end{aligned}$$

These equations can be written in matrix form as

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{-t/2}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2e^{-t/6}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using the initial condition $x(0) = 5$, $y(0) = 2$, we find $c_1 = 2$, $c_2 = 3$. Geometrically, these equations are the parametric equations of a curve (trajectory of the DE) in the xy-plane (phase plane of the DE). As $t \to \infty$ we have $(x(t), y(t)) \to (0, 0)$. The constant solution $x(t) = y(t) = 0$ is called an **equilibrium solution** of our system. This solution is said to be **asymptotically stable** if the general solution converges to it as $t \to \infty$. A system is called **stable** if the trajectories are all bounded as $t \to \infty$.

Our system can be written in matrix form as $\frac{dX}{dt} = AX$ where

$$A = \begin{bmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{bmatrix} X.$$

The 2×2 matrix A is called the matrix of the system. The polynomial

$$r^2 - \operatorname{tr}(A)r + \det(A) = r^2 + \frac{2}{3}r + \frac{1}{12}$$

where $\operatorname{tr}(A)$ is the trace of A (sum of diagonal entries) and $\det(A)$ is the determinant of A is called the **characteristic polynomial** of A. Notice that this polynomial is the characteristic polynomial of the differential equation for x. The equations

$$A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{-1}{6} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

identify $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as eigenvectors of A with eigenvalues $-1/2$ and $-1/6$ respectively. More generally, a non-zero column vector X is an **eigenvector** of a square matrix A with **eigenvalue** r if $AX = rX$ or , equivalently, $(rI - A)X = 0$. The latter is a homogeneous system of linear equations with coefficient matrix $rI - A$. Such a system has a non-zero solution if and only if $\det(rI - A) = 0$. Notice that

$$\det(rI - A) = r^2 - (a + d)r + ad - bc$$

is the characteristic polynomial of A.

If, in the above mixing problem, brine at a concentration of 1/2 kg/L was pumped into tank A instead of pure water the system would be

$$\begin{aligned} \frac{dx}{dt} &= \frac{-1}{3}x + \frac{1}{12}y + 3, \\ \frac{dy}{dt} &= \frac{1}{3}x - \frac{1}{3}y, \end{aligned}$$

a non-homogeneous system. Here an equilibrium solution would be $x(t) = a, y(t) = b$ where (a, b) was a solution of

$$\begin{aligned} \frac{-1}{3}x + \frac{1}{12}y &= -3, \\ \frac{1}{3}x - \frac{1}{3}y &= 0. \end{aligned}$$

In this case $a = b = 12$. The variables $x^* = x - 12, y^* = y - 12$ then satisfy the homogeneous system

$$\begin{aligned} \frac{dx^*}{dt} &= \frac{-1}{3}x^* + \frac{1}{12}y^*, \\ \frac{dy^*}{dt} &= \frac{1}{3}x^* - \frac{1}{3}y^*. \end{aligned}$$

Solving this system as above for x^*, y^* we get $x = x^* + 12, y = y^* + 12$ as the general solution for x, y.

2 (2×2) System of Linear Equations

We now describe the solution of the system $\frac{dX}{dt} = AX$ for an arbitrary 2×2 matrix A. In practice, one can use the elimination method or the eigenvector method but we shall use the eigenvector method as it gives an explicit description of the solution. There are three main cases depending on whether the discriminant

$$\Delta = \operatorname{tr}(A)^2 - 4\det(A)$$

of the characteristic polynomial of A is > 0, < 0, $= 0$.

2.1 Case 1: $\Delta > 0$

In this case the roots r_1, r_2 of the characteristic polynomial are real and unequal, say $r_1 < r_2$. Let P_i be an eigenvector with eigenvalue r_i. Then P_1 is not a scalar multiple of P_2 and so the matrix P with columns P_1, P_2 is invertible. After possibly replacing P_2 by $-P_2$, we can assume that $\det(P) > 0$. The equation

$$AP = P \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

shows that

$$P^{-1}AP = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}.$$

If we make the change of variable $X = PU$ with $U = \begin{bmatrix} u \\ v \end{bmatrix}$, our system becomes

$$P\frac{dU}{dt} = APU \quad \text{or} \quad \frac{dU}{dt} = P^{-1}APU.$$

Hence, our system reduces to the uncoupled system

$$\frac{du}{dt} = r_1 u, \quad \frac{dv}{dt} = r_2 v$$

which has the general solution $u = c_1 e^{r_1 t}$, $v = c_2 e^{r_2 t}$. Thus the general solution of the given system is

$$X = PU = uP_1 + vP_2 = c_1 e^{r_1 t} P_1 + c_2 e^{r_2 t} P_2.$$

Since $\operatorname{tr}(A) = r_1 + r_2$, $\det(A) = r_1 r_2$, we see that $x(t), y(t) = (0, 0)$ is an asymptotically stable equilibrium solution if and only if $\operatorname{tr}(A) < 0$ and $\det(A) > 0$. The system is unstable if $\det(A) < 0$ or $\det(A) \geq 0$ and $\operatorname{tr}(A) \geq 0$.

2.2 Case 2: $\Delta < 0$

In this case the roots of the characteristic polynomial are complex numbers

$$r = \alpha \pm i\omega = \operatorname{tr}(A)/2 \pm i\sqrt{\Delta/4}.$$

The corresponding eigenvectors of A are (complex) scalar multiples of

$$\begin{bmatrix} 1 \\ \sigma \pm i\tau \end{bmatrix}$$

where $\sigma = (\alpha - a)/b$, $\tau = \omega/b$. If X is a real solution we must have $X = V + \overline{V}$ with

$$V = \frac{1}{2}(c_1 + ic_2)e^{\alpha t}(\cos(\omega t) + i\sin(\omega t))\begin{bmatrix} 1 \\ \sigma + i\tau \end{bmatrix}.$$

It follows that

$$X = e^{\alpha t}(c_1\cos(\omega t) - c_2\sin(\omega t))\begin{bmatrix} 1 \\ \sigma \end{bmatrix} + e^{\alpha t}(c_1\sin(\omega t) + c_2\cos(\omega t))\begin{bmatrix} 0 \\ \tau \end{bmatrix}.$$

The trajectories are spirals if $\mathrm{tr}(A) \neq 0$ and ellipses if $\mathrm{tr}(A) = 0$. The system is asymptotically stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) > 0$.

2.3 Case 3: $\Delta = 0$

Here the characteristic polynomial has only one root r. If $A = rI$ the system is

$$\frac{dx}{dt} = rx, \quad \frac{dy}{dt} = ry.$$

which has the general solution $x = c_1e^{rt}$, $y = c_2e^{rt}$. Thus the system is asymptotically stable if $\mathrm{tr}(A) < 0$, stable if $\mathrm{tr}(A) = 0$ and unstable if $\mathrm{tr}(A) > 0$.

Now suppose $A \neq rI$. If P_1 is an eigenvector with eigenvalue r and P_2 is chosen with $(A - rI)P_1 \neq 0$, the matrix P with columns P_1, P_2 is invertible and

$$P^{-1}AP = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}.$$

Setting as before $X = PU$ we get the system

$$\frac{du}{dt} = ru + v, \quad \frac{dv}{dt} = rv$$

which has the general solution $u = c_1e^{rt} + c_2te^{rt}$, $v = c_2e^{rt}$. Hence the given system has the general solution

$$X = uP_1 + vP_2 = (c_1e^{rt} + c_2te^{rt})P_1 + c_2e^{rt}P_2.$$

The trajectories are asymptotically stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) \geq 0$.

A non-homogeneous system $\frac{dX}{dt} = AX + B$ having an equilibrium solution $x(t) = x_1, y(t) = y_1$ can be solved by introducing new variables $x^* = x - x_1, y^* = y - y_1$. Since $AX^* + B = 0$ we have

$$\frac{dX^*}{dt} = AX^*,$$

a homogeneous system which can be solved as above.