McGill University
Math 325A: Differential Equations

LECTURE 20 : LAPLACE TRANSFORMS (II)

(Text: Chap. 7)

1 Introduction

In this lecture we will, by using examples, show how to use Laplace transforms in solving differential equations with constant coefficients.

2 Example 1

Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

2.1 Step 1

Let $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Taking Laplace transforms of the DE, we get

$$(s^2 + s + 1)Y(s) - s = \frac{1}{s^2 + 1}.$$

2.2 Step 2

Solving for $Y(s)$, we get

$$Y(s) = \frac{s}{s^2 + s + 1} + \frac{1}{(s^2 + s + 1)(s^2 + 1)}.$$

2.3 Step 3

Finding the inverse laplace transform.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2 + s + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2 + s + 1)(s^2 + 1)}\right\}.$$

Since

$$\frac{s}{s^2 + s + 1} = \frac{s}{(s + 1/2)^2 + 3/4} = \frac{s + 1/2}{(s + 1/2)^2 + (\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s + 1/2)^2 + (\sqrt{3}/2)^2}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 + s + 1}\right\} = e^{-t/2}\cos(\sqrt{3}\ t/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

Multiplying both sides by $(s^2+1)(s^2+s+1)$ and collecting terms, we find

$$1 = (A+C)s^3 + (B+C+D)s^2 + (A+C+D)s + B + D.$$

Equating coefficients, we get $A+C=0$, $B+C+D=0$, $A+C+D=0$, $B+D=1$, from which we get $A=B=1$, $C=-1$, $D=0$, so that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2), \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

we obtain

$$y(t) = 2e^{-t/2}\cos(\sqrt{3}\ t/2) - \cos(t).$$

3 Example 2

As we have known, a higher order DE can be reduced to a system of DE's. Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= -2x + y, \\ \frac{dy}{dt} &= x - 2y \end{aligned}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$.

3.1 Step 1

Taking Laplace transforms the system becomes

$$\begin{aligned} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{aligned}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$.

3.2 Step 2

Solving for $X(s)$, $Y(s)$. The above linear system of equations can be written in the form:

$$\begin{aligned} (s+2)X(s) - Y(s) &= 1, \\ -X(s) + (s+2)Y(s) &= 2. \end{aligned}$$

The determinant of the coefficient matrix is $s^2+4s+3 = (s+1)(s+3)$. Using Cramer's rule we get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

3.3 Step 3

Finding the inverse Laplace transform. Since

$$\frac{s+4}{(s+1)(s+3)} = \frac{3/2}{s+1} - \frac{1/2}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3/2}{s+1} + \frac{1/2}{s+3},$$

we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

The Laplace transform reduces the solution of differential equations to a partial fractions calculation. If $F(s) = P(s)/Q(s)$ is a ratio of polynomials with the degree of $P(s)$ less than the degree of $Q(s)$ then $F(s)$ can be written as a sum of terms each of which corresponds to an irreducible factor of $Q(s)$. Each factor $Q(s)$ of the form $s - a$ contributes the terms

$$\frac{A_1}{s-a} + \frac{A_1}{(s-a)^2} + \cdots + \frac{A_r}{(s-a)^r}$$

where r is the multiplicity of the factor $s-a$. Each irreducible quadratic factor s^2+as+b contributes the terms

$$\frac{A_1 s + B_1}{s^2+as+b} + \frac{A_2 s + B_2}{(s^2+as+b)^2} + \cdots + \frac{A_r s + B_r}{(s^2+as+b)^r}$$

where r is the degree of multiplicity of the factor $s^2 + as + b$.