

LECTURE 19 : LAPLACE TRANSFORMS (I)

(Text: Chap. 7)

1 Introduction

We begin our study of the Laplace Transform with a motivating example. This example illustrates the type of problem that the Laplace transform was designed to solve.

A mass-spring system consisting of a single steel ball is suspended from the ceiling by a spring. For simplicity, we assume that the mass and spring constant are 1. Below the ball we introduce an electromagnet controlled by a switch. Assume that, when on, the electromagnet exerts a unit force on the ball. After the ball is in equilibrium for 10 seconds the electromagnet is turned on for 2π seconds and then turned off. Let $y = y(t)$ be the downward displacement of the ball from the equilibrium position at time t . To describe the motion of the ball using techniques previously developed we have to divide the problem into three parts: (I) $0 \leq t < 10$; (II) $10 \leq t < 10 + 2\pi$; (III) $10 + 2\pi \leq t$. The initial value problem determining the motion in part I is

$$y'' + y = 0, \quad y(0) = y'(0) = 0.$$

The solution is $y(t) = 0$, $0 \leq t < 10$. Taking limits as $t \rightarrow 10$ from the left, we find $y(10) = y'(10) = 0$. The initial value problem determining the motion in part II is

$$y'' + y = 1, \quad y(10) = y'(10) = 0.$$

The solution is $y(t) = 1 - \cos(t - 10)$, $10 \leq t < 10 + 2\pi$. Taking limits as $t \rightarrow 10 + 2\pi$ from the left, we get $y(10 + 2\pi) = y'(10 + 2\pi) = 0$. The initial value problem for the last part is

$$y'' + y = 0, \quad y(10 + 2\pi) = y'(10 + 2\pi) = 0$$

which has the solution $y(t) = 0$, $10 + 2\pi \leq t$. Putting all this together, we have

$$y(t) = \begin{cases} 0, & 0 \leq t < 10, \\ 1 - \cos(t - 10), & 10 \leq t < 10 + 2\pi, \\ 0, & 10 + 2\pi \leq t. \end{cases}$$

The function $y(t)$ is continuous with continuous derivative

$$y'(t) = \begin{cases} 0, & 0 \leq t < 10, \\ \sin(t - 10), & 10 \leq t < 10 + 2\pi, \\ 0, & 10 + 2\pi \leq t. \end{cases}$$

However the function $y'(t)$ is not differentiable at $t = 10$ and $t = 10 + 2\pi$. In fact

$$y''(t) = \begin{cases} 0, & 0 \leq t < 10, \\ \cos(t - 10), & 10 < t < 10 + 2\pi, \\ 0, & 10 + 2\pi < t. \end{cases}$$

The left hand and right hand limits of $f''(t)$ at $t = 10$ are 0 and 1 respectively. At $t = 10 + 2\pi$ they are 1 and 0. If we extend $y''(t)$ by using the left-hand or righthand limits at 10 and $10 + 2\pi$ the resulting function is not continuous. Such a function with only jump discontinuities is said to be **piecewise continuous**. If we try to write the differential equation of the system we have

$$y'' + y = f(t) = \begin{cases} 0, & 0 \leq t < 10, \\ 1, & 10 \leq t < 10 + 2\pi, \\ 0, & 10 + 2\pi \leq t. \end{cases}$$

Here $f(t)$ is piecewise continuous and any solution would also have y'' piecewise continuous. By a solution we mean any function $y = y(t)$ satisfying the DE for those t not equal to the points of discontinuity of $f(t)$. In this case we have shown that a solution exists with $y(t)$, $y'(t)$ continuous. In the same way, one can show that in general such solutions exist using the fundamental theorem.

What we want to describe now is a mechanism for doing such problems without having to divide the problem into parts. This mechanism is the Laplace transform.

2 Laplace Transform

2.1 Definition

Let $f(t)$ be a function defined for $t \geq 0$. The function $f(t)$ is said to be **piecewise continuous** if

- (1) $f(t)$ converges to a finite limit $f(0_+)$ as $t \rightarrow 0_+$
- (2) for any $c > 0$, the left and right hand limits $f(c_-)$, $f(c_+)$ of $f(t)$ at c exist and are finite.
- (3) $f(c_-) = f(c_+) = f(c)$ for every $c > 0$ except possibly a finite set of points or an infinite sequence of points converging to $+\infty$. Thus the only points of discontinuity of $f(t)$ are jump discontinuities.

The function is said to be **normalized** if $f(c) = f(c_+)$ for every $c \geq 0$.

The Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is the function of a new variable s defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{N \rightarrow +\infty} \int_0^N e^{-st} f(t) dt.$$

An important class of functions for which the integral converges are the functions of exponential order. The function $f(t)$ is said to be of **exponential order** if there are constants a, M such that

$$|f(t)| \leq M e^{at}$$

for all t . the solutions of constant coefficient homogeneous DE's are all of exponential order. The convergence of the improper integral follows from

$$\int_0^N |e^{-st} f(t)| dt \leq M \int_0^N e^{-(s-a)t} dt = \frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a}$$

which shows that the improper integral converges absolutely when $s > a$. It shows that $F(s) \rightarrow 0$ as $s \rightarrow \infty$. The calculation also shows that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

for $s > a$. Setting $a = 0$, we get $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

2.2 Basic Properties and Formulas

The above holds when $f(t)$ is complex valued and $s = \sigma + i\tau$ is complex. The integral exists in this case for $\sigma > a$. For example, this yields

$$\mathcal{L}\{e^{it}\} = \frac{1}{s-i}, \quad \mathcal{L}\{e^{-it}\} = \frac{1}{s+i}.$$

2.2.1 Linearity of the transform

$$\mathcal{L}\{af(t) + bf(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Using this linearity property of the Laplace transform and using $\sin(t) = (e^{it} - e^{-it})/2i$, $\cos(t) = (e^{it} + e^{-it})/2$, we find

$$\begin{aligned} \mathcal{L}\{\sin(bt)\} &= \frac{1}{2i} \left(\frac{1}{s-bi} - \frac{1}{s+bi} \right) = \frac{b}{s^2 + b^2}, \\ \mathcal{L}\{\cos(bt)\} &= \frac{1}{2} \left(\frac{1}{s-bi} + \frac{1}{s+bi} \right) = \frac{s}{s^2 + b^2}, \end{aligned}$$

for $s > 0$.

2.2.2 Formula (I)

The following two identities follow from the definition of the Laplace transform after a change of variable.

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a), \quad \mathcal{L}\{f(bt)\}(s) = \frac{1}{b}\mathcal{L}\{f(t)\}(s/b).$$

Using the first of these formulas, we get

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2 + b^2}, \quad \mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}.$$

2.2.3 Formula (II)

The next formula will allow us to find the Laplace transform for all the functions that are annihilated by a constant coefficient differential operator.

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}(s).$$

For $n = 1$ this follows from the definition of the Laplace transform on differentiating with respect s and taking the derivative inside the integral. The general case follows by induction. For example, using this formula, we obtain using $f(t) = 1$

$$\mathcal{L}\{t^n\}(s) = -\frac{d^n}{ds^n} \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

With $f(t) = \sin(t)$ and $f(t) = \cos(t)$ we get

$$\begin{aligned} \mathcal{L}\{t \sin(bt)\}(s) &= -\frac{d}{ds} \frac{b}{s^2 + b^2} = \frac{2bs}{(s^2 + b^2)^2}, \\ \mathcal{L}\{t \cos(bt)\}(s) &= -\frac{d}{ds} \frac{s}{s^2 + b^2} = \frac{s^2 - b^2}{(s^2 + b^2)^2} = \frac{1}{s^2 + b^2} - \frac{2b^2}{(s^2 + b^2)^2}. \end{aligned}$$

2.2.4 Formula (III)

The next formula shows how to compute the Laplace transform of $f'(t)$ in terms of the Laplace transform of $f(t)$.

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

This follows from

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t) dt = e^{-st} f(t)|_0^\infty + s \int_0^\infty e^{-st} f(t) dt = s \int_0^\infty e^{-st} f(t) dt - f(0)$$

since $e^{-st} f(t)$ converges to 0 as $t \rightarrow +\infty$ in the domain of definition of the Laplace transform of $f(t)$. To ensure that the first integral is defined, we have to assume $f'(t)$ is piecewise continuous. Repeated applications of this formula give

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

The following theorem is important for the application of the Laplace transform to differential equations.

3 Inverse Laplace Transform

3.1 Theorem:

If $f(t)$, $g(t)$ are normalized piecewise continuous functions of exponential order then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \implies f = g.$$

3.2 Definition

If $F(s)$ is the Laplace of the normalized piecewise continuous function $f(t)$ of exponential order then $f(t)$ is called the **inverse Laplace transform** of $F(s)$. This is denoted by

$$F(s) = \mathcal{L}^{-1}\{f(t)\}.$$

Note that the inverse Laplace transform is also linear. Using the Laplace transforms we found for $t \sin(bt)$, $t \cos(bt)$ we find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + b^2)^2}\right\} = \frac{1}{2b} t \sin(bt), \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^2 + b^2)^2}\right\} = \frac{1}{2b^3} \sin(bt) - \frac{1}{2b^2} t \cos(bt).$$