McGill University
Math 325A: Differential Equations

LECTURE 15: SOME APPLICATIONS OF SECOND ORDER DE'S

HIGHER ORDER DIFFERENTIAL EQUATIONS (VII)

(Text: Chap. 4)

1 (*) Vibration System

We now give an application of the theory of second order DE's to the description of the motion of a simple mass-spring mechanical system with a damping device. We assume that the damping force is proportional to the velocity of the mass. If there are no external forces we obtain the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

where $x = x(t)$ is the displacement from equilibrium at time t of the mass of $m > 0$ units, $b \geq 0$ is the damping constant and $k > 0$ is the spring constant. In operator form with $D = \frac{d}{dt}$ this DE is, after normalizing,

$$\left(D^2 + \frac{b}{m}D + \frac{k}{m}\right)(x) = 0.$$

The characteristic polynomial $r^2 + (b/m)r + k/m$ has discriminant

$$\Delta = (b^2 - 4km)/m^2.$$

If $b^2 < 4km$ we have $\Delta < 0$ and the characteristic polynomial factorizes in the form $(r+b/2m)^2+\omega^2$ with

$$\omega = \sqrt{4km - b^2}/2m = \sqrt{\frac{k}{m} - (b/2m)^2}.$$

In this case the characteristic polynomial has complex roots $-b/2m \pm i\omega$ and the general solution of the DE is

$$y = e^{-bt/2m}(A\cos(\omega t) + B\sin(\omega t) = Ce^{-bt/2m}\sin(\omega t + \theta)$$

where $C = \sqrt{A^2 + B^2}$ and $0 \leq \theta \leq 2\pi$ defined by $\cos(\theta) = A/C$, $\sin(\theta) = B/C$. The angle θ is called the **phase**. In this case we see that the mass oscillates with **frequency** $\omega/2\pi$ and decreasing amplitude. If $b = 0$ there is no damping and the mass oscillates with frequency $\omega/2\pi$ and constant amplitude; such motion is called **simple harmonic**.

If $b^2 \geq 4km$ we have $\Delta \geq 0$ and so the characteristic polynomial has real roots

$$r_1 = -b/2m + \sqrt{b^2 - 4km}/2m, \quad r_2 = -b/2m - \sqrt{b^2 - 4km}/2m$$

which are both negative. If $r_1 = r_2 = r$ the general solution of our DE is

$$y = Ae^{rt} + Bte^{rt}$$

and if $r_1 \neq r_2$ the general solution is

$$y = Ae^{r_1t} + Be^{r_2t}.$$

In both cases $y \to 0$ as $t \to \infty$. In the second case we have what is called **over damping** and in the first case the over damping is said to be **critical**. In each the mass returns to its equilibrium position without oscillations.

Suppose now that our mass-spring system is subject to an external force so that our DE now becomes

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t).$$

The function $F(t)$ is called the **forcing function** and measures the magnitude and direction of the external force. We consider the important special case where the forcing function is harmonic

$$F(f) = F_0\cos(\gamma t), \quad F_0 > 0 \text{ a constant.}$$

We also assume that we have under-damping with damping constant $b > 0$. In this case the DE has a particular solution of the form

$$y_p = A_1\cos(\gamma t) + A_2\sin(\gamma t).$$

Substituting the the DE and simplifying, we get

$$((k - m\gamma^2)A_1 + b\gamma A_2)\cos(\gamma t) + (-b\gamma A_1 + (k - m\gamma^2)A_2)\sin(\gamma t) = F_0\cos(\gamma t).$$

Setting the corresponding coefficients on both sides equal, we get

$$\begin{aligned} (k - m\gamma^2)A_1 + b\gamma A_2 &= F_0, \\ -b\gamma A_1 + (k - m\gamma^2)A_2 &= 0. \end{aligned}$$

Solving for A_1, A_2 we get

$$A_1 = \frac{F_0(k - m\gamma^2)}{(k - m\gamma^2)^2 + b^2\gamma^2}, \quad A_2 = \frac{F_0 b\gamma}{(k - m\gamma^2)^2 + b^2\gamma^2}$$

and

$$\begin{aligned} y_p &= \frac{F_0}{(k - m\gamma^2)^2 + b^2\gamma^2}((k - m\gamma^2)\cos(\gamma t) + b\gamma\sin(\gamma t)) \\ &= \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}\sin(\gamma t + \phi). \end{aligned}$$

The general solution of our DE is then

$$y = Ce^{-bt/2m}\sin(\omega t + \theta) + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}\sin(\gamma t + \phi).$$

Because of damping the first term tends to zero and is called the **transient** part of the solution. The second term, the **steady-state** part of the solution, is due to the presence of the forcing function $F_0\cos(\gamma t)$. It is harmonic with the same frequency $\gamma/2\pi$ but is out of phase with it by an angle $\phi - \pi/2$. The ratio of the magnitudes

$$M(\gamma) = \frac{1}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$$

is called the **gain** factor. The graph of the function $M(\gamma)$ is called the **resonance curve**. It has a maximum of

$$\frac{1}{b\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

when $\gamma = \gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$. The frequency $\gamma_r/2\pi$ is called the **resonance frequency** of the system. When $\gamma = \gamma_r$ the system is said to be in resonance with the external force. Note that $M(\gamma_r)$ gets arbitrarily large as $b \to 0$. We thus see that the system is subject to large oscillations if the damping constant is very small and the forcing function has a frequency near the resonance frequency of the system.

The above applies to a simple LRC electrical circuit where the differential equation for the current I is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + I/C = F(t)$$

where L is the inductance, R is the resistance and C is the capacitance. The resonance phenomenon is the reason why we can send and receive and amplify radio transmissions sent simultaneously but with different frequencies.