McGill University
Math 325A: Differential Equations

LECTURE 13 : FINDING A PARTICULAR SOLUTION FOR INHOMOGENEOUS EQUATION

HIGHER ORDER DIFFERENTIAL EQUATIONS (V)

(Text: Chap. 4, 6)

1 Introduction

Variation of parameters is method for producing a particular solution of a special kind for the general linear DE in normal form

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$$

from a fundamental set $y_1, y_2, \dots, y_n$ of solutions of the associated homogeneous equation.

2 The Differential Operator for Equations with Constant Coefficients

Given

$$L(y) = P(D)(y) = (a_0 D^{(n)} + a_1 D^{(n-1)} + \cdots + a_n D)y = b(x).$$

Assume that the inhomogeneous term $b(x)$ is a solution of linear equation:

$$Q(D)\big(b(x)\big) = 0.$$

Then we can transform the original inhomogeneous equation to the homogeneous equation by applying the differential operator $Q(D)$ to its both sides,

$$\Phi(D)(y) = Q(D)P(D)(y) = 0.$$

The operator $Q(D)$ is called the *Annihilator*. The above method is also called the *Annihilator Method*.

Example 1. Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = x + e^x, \quad y(0) = 1, y'(0) = y''(0) = 0.$$

This DE is non-homogeneous. The associated homogeneous equation was solved in the previous lecture. Note that in this example, In the inhomogeneous term $b(x) = x + e^x$ is in the kernel of $Q(D) = D^2(D-1)$. Hence, we have

$$D^2(D-1)^2((D-1)^2 + 4)(y) = 0$$

which yields $y = Ax + B + Cxe^x + c_1e^x + c_2e^x \cos(2x) + c_3e^x \sin(2x)$. This shows that there is a particular solution of the form $y_P = Ax + B + Cxe^x$ which is obtained by discarding the terms in the solution space of the associated homogeneous DE. Substituting this in the original DE we get

$$y''' - 3y'' + 7y' - 5y = 7A - 5B - 5Ax - Ce^x$$

which is equal to $x + e^x$ if and only if $7A - 5B = 0$, $-5A = 1$, $-C = 1$ so that $A = -1/5$, $B = -7/25$, $C = -1$. Hence the general solution is

$$y = c_1e^x + c_2e^x \cos(2x) + c_3e^x \sin(2x) - x/5 - 7/25 - xe^x.$$

To satisfy the initial condition $y(0) = 0, y'(0) = y''(0) = 0$ we must have

$$\begin{aligned} c_1 + c_2 &= 32/25, \\ c_1 + c_2 + 2c_3 &= 6/5, \\ c_1 - 3c_2 + 4c_3 &= 2 \end{aligned}$$

which has the solution $c_1 = 3/2, c_2 = -11/50, c_3 = -1/25$.

It is evident that if the function $b(x)$ can not be eliminated by any linear operator $Q(D)$, the annihilator method will not applicable.

3 The Method of Variation of Parameters

In this method we try for a solution of the form

$$y_P = C_1(x)y_1 + C_2(x)y_2 + \cdots + C_n(x)y_n.$$

Then $y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n' + C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n$ and we impose the condition

$$C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n = 0.$$

Then $y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n'$ and hence

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'' + C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n'.$$

Again we impose the condition $C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' = 0$ so that

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'.$$

We do this for the first $n-1$ derivatives of y so that for $1 \le k \le n-1$

$$y_P^{(k)} = C_1(x)y_1^{(k)} + C_2(x)y_2^{(k)} + \cdots C_n(x)y_n^{(k)},$$

$$C_1'(x)y_1^{(k)} + C_2'(x)y_2^{(k)} + \cdots + C_n'(x)y_n^{(k)} = 0.$$

Now substituting $y_P, y_P', \ldots, y_P^{(n-1)}$ in $L(y) = b(x)$ we get

$$C_1(x)L(y_1) + C_2(x)L(y_2) + \cdots + C_n(x)L(y_n) + C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x).$$

But $L(y_i) = 0$ for $1 \le k \le n$ so that

$$C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x).$$

We thus obtain the system of n linear equations for $C_1'(x), \ldots, C_n'(x)$

$$\begin{aligned} C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n &= 0, \\ C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' &= 0, \\ &\vdots \\ C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} &= b(x). \end{aligned}$$

If we solve this system using Cramer's Rule and integrate, we find

$$C_i(x) = \int_{x_0}^{x} (-1)^{n+i} b(t) \frac{W_i}{W} dt$$

where $W = W(y_1, y_2, \ldots, y_n)$ and $W_i = W(y_1 \ldots, \hat{y}_i, \ldots, y_n)$ where the $\hat{}$ means that y_i is omitted. Note that the particular solution y_P found in this way satisfies

$$y_P(x_0) = y_P'(x_0) = \cdots = y_P^{(n-1)} = 0.$$

The point x_0 is any point in the interval of continuity of the $a_i(x)$ and $b(x)$. Note that y_P is a linear function of the function $b(x)$.

Example 2. Find the general solution of $y'' + y = 1/x$ on $x > 0$.

The general solution of $y'' + y = 0$ is $y = c_1 \cos(x) + c_2 \sin(x)$. Using variation of parameters with $y_1 = \cos(x)$, $y_2 = \sin(x)$, $b(x) = 1/x$ and $x_0 = 1$, we have $W = 1$, $W_1 = \sin(x)$, $W_2 = \cos(x)$ and we obtain the particular solution $y_p = C_1(x)\cos(x) + C_2(x)\sin(x)$ where

$$C_1(x) = -\int_1^x \frac{\sin(t)}{t} dt, \quad C_2(x) = \int_1^x \frac{\cos(t)}{t} dt.$$

The general solution of $y'' + y = 1/x$ on $x > 0$ is therefore

$$y = c_1 \cos(x) + c_2 \sin(x) - \left(\int_1^x \frac{\sin(t)}{t} dt \right) \cos(x) + \left(\int_1^x \frac{\cos(t)}{t} dt \right) \sin(x).$$

When applicable, the annihilator method is easier as one can see from the DE $y'' + y = e^x$. Here it is immediate that $y_p = e^x/2$ is a particular solution while variation of parameters gives

$$y_p = -\left(\int_0^x e^t \sin(t) dt \right) \cos(x) + \left(\int_0^x e^t \cos(t) dt \right) \sin(x).$$

The integrals can be evaluated using integration by parts:

$$\begin{aligned} \int_0^x e^t \cos(t) dt &= e^x \cos(x) - 1 + \int_0^x e^t \sin(t) dt \\ &= e^x \cos(x) + e^x \sin(x) - 1 - \int_0^x e^t \cos(t) dt \end{aligned}$$

which gives

$$\int_0^x e^t \cos(t) dt = \Big[e^x \cos(x) + e^x \sin(x) - 1 \Big]/2$$

$$\int_0^x e^t \sin(t) dt = e^x \sin(x) - \int_0^x e^t \cos(t) dt = \Big[e^x \sin(x) - e^x \cos(x) + 1 \Big]/2$$

so that after simplification $y_p = e^x/2 - \cos(x)/2 - \sin(x)/2$.