Perturbed Haar function expansions

Michael Wilson
Department of Mathematics
University of Vermont
Burlington, Vermont 05405

Abstract

The Haar functions $\{\phi_{(I)}\}_{I\in\mathcal{D}_1}$ form a complete orthonormal system for $L^2(\mathbf{R})$ and an unconditional basis for $L^p(\mathbf{R})$ $(1 < p < \infty)$: $\forall f \in L^p(\mathbf{R})$,

$$f = \sum_I \langle f, \phi_{(I)}\rangle \phi_{(I)}, \tag{1}$$

where $\mathcal{D}_1$ is the set of dyadic intervals, $\langle f, g\rangle := \int f\,\overline{g}\,dx$ is the usual inner product, and the series converges unconditionally in L^p. The Haar functions are readily generalized to a family $\{\phi_{(I),k}\}_{\substack{I\in\mathcal{D}_d \\ 1\leq k<2^d}}$ defined on $\mathbf{R}^d$, with $\mathcal{D}_d$ being the dyadic cubes, for which the analogous convergence result holds.

We explore the stability of (1) when the $\phi_{(I),k}$s are subjected to arbitrary, close-to-the-identity, (local) affine changes of variable. In one dimension this means replacing each $\phi_{(I)}$ by, respectively, $\widetilde{\phi_{(I)}}$ and $\phi^*_{(I)}$,

$$\begin{aligned}\widetilde{\phi_{(I)}}(x) &:= \phi_{(I)}(\widetilde{\alpha}_I(x - x_I + \widetilde{y}_I\ell(I)) + x_I) \\ \phi^*_{(I)}(x) &:= \phi_{(I)}(\alpha^*_I(x - x_I + y^*_I\ell(I)) + x_I),\end{aligned}$$

where x_I is I's center, $\ell(I)$ is its length, and $\{\widetilde{\alpha}_I\}_{I\in\mathcal{D}_1}$, $\{\alpha^*_I\}_{I\in\mathcal{D}_1}$, $\{\widetilde{y}_I\}_{I\in\mathcal{D}_1}$, and $\{y^*_I\}_{I\in\mathcal{D}_1}$ are sets of real numbers satisfying

$$\sup_{I\in\mathcal{D}_1} \max\left(|1-\widetilde{\alpha}_I| + |\widetilde{y}_I|, |1-\alpha^*_I| + |y^*_I|\right) \leq \eta$$

for a fixed $0 \leq \eta < 1/2$. We show that, for all $f \in L^2(\mathbf{R})$,

$$\left\| f - \sum_I \langle f, \widetilde{\phi}_{(I)}\rangle \phi^*_{(I)} \right\|_2 \leq C\eta^{1/2}\|f\|_2,$$

with C an absolute constant, and

$$\left\| f - \sum_I \langle f, \widetilde{\phi}_{(I)}\rangle \phi_{(I)} \right\|_p \leq C\eta^{1/2}\|f\|_p$$

for all $f \in L^p(\mathbf{R})$, with $C = C(p)$ for $2 < p < \infty$; and the analogous results, with more complicated notations, in $L^p(\mathbf{R}^d)$ $(2 \leq p < \infty)$. (The "missing" $*$ in the second inequality is not a typo.)