Finding Traveling Wave Solutions to Lattice Differential Equations

R.P. Wilds & A.R. Humphries

wilds@math.mcgill.ca
http://www.roywilds.ca

McGill University

Lattice Differential Equations

Translationally Invariant, Finite Interaction, Lattice Differential Equation (LDE)

$$\dot{v}_i(t) = \sum_{j=-n}^{n} H_j(v_{i+j}(t)) \qquad i, j, n \in \mathbb{Z}, t \in \mathbb{R}$$

- $v_i \in \mathbb{C}^m$
- Can relax above constraints

Case Problem

Discrete Nagumo Equation

$$\dot{v}_i(t) = v_{i+1}(t) - 2v_i(t) + v_{i-1}(t) - \beta f(v_i(t))$$

- $\beta \in \mathbb{R}^+$ determines the relative strength between the coupling $v_{i+1} - 2v_i + v_{i-1}$ and the forcing $f(v_i)$
- Bistable nonlinearity $f(v) = v(v-a)(v-1)$ with $a \in (0,1)$.

Traveling Wave Solutions

- Introduce a Traveling Wave (TW) ansatz:
- Let $\zeta = i - ct$, $\varphi(\zeta) = v_i(t)$

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$$-c\varphi'(\zeta) = \varphi(\zeta+1) - 2\varphi(\zeta) + \varphi(\zeta-1) - \beta f(\varphi(\zeta))$$

- $i \in \mathbb{Z}$. However, $\zeta \in \mathbb{R}$ since $t \in \mathbb{R}$.

Connecting Orbits

$$\begin{aligned} -c\varphi'(\zeta) &= \varphi(\zeta+1) - 2\varphi(\zeta) + \varphi(\zeta-1) \\ &- \beta\varphi(\zeta)[\varphi(\zeta)-a][\varphi(\zeta)-1] \end{aligned}$$

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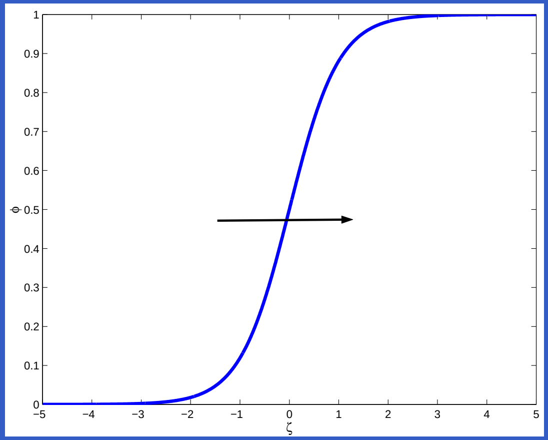
$$

Constant solutions

- Stable: $\varphi(\zeta) = 0, 1$
- Unstable: $\varphi(\zeta) = a$



We look for heteroclinic orbits which connect 0 to 1.

Advances/Delays

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- Delays/Advances: $\varphi(\zeta-1)$,$\varphi(\zeta+1)$.
- To know the dynamics in $\zeta \in [-1,1]$ need to know the solution over $\zeta \in [-2,2]$.

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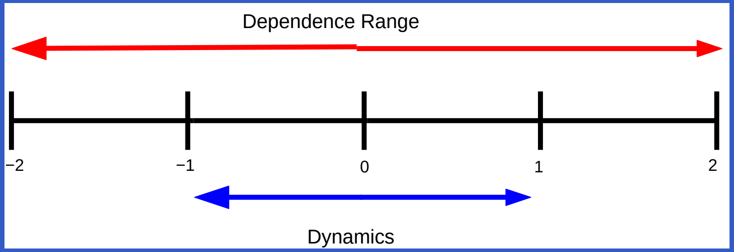

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$$\begin{aligned} \varphi(\zeta \to -\infty) &\approx 0 \\ \varphi(\zeta \to +\infty) &\approx 1 \end{aligned}$$

Can obtain analytical approximation to solution when $\varphi \approx 0, 1$ by linearizing the dynamics

Linear MFDEs

$$
\begin{aligned}
-c\varphi'(\zeta) &= \varphi(\zeta+1) - 2\varphi(\zeta) + \varphi(\zeta-1) \\
&- \beta a \varphi(\zeta)
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- Obtain fundamental modes $e^{\lambda\zeta}$ by solving characteristic equation.
- Yields approximations to the solution where the linearization is valid.

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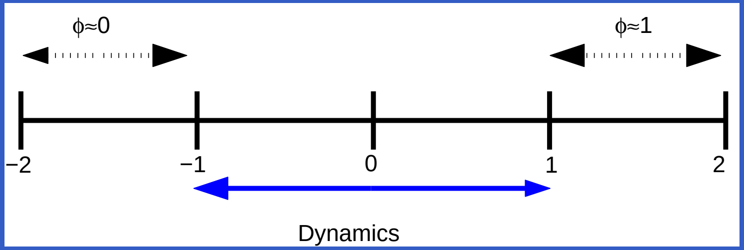

Continuum Formulation

$$\dot{v}_i(t) = v_{i+1}(t) - 2v_i(t) + v_{i-1}(t) - \beta f(v_i(t))$$

- Let $i \to \eta \in \mathbb{R}$, take on non-integer values too.

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- Let $i \to \eta \in \mathbb{R}$, take on non-integer values too.

$$\dot{v}(\eta, t) = v(\eta + 1, t) - 2v(\eta, t) + v(\eta - 1, t) - f(v(\eta, t))$$

- Any solution with η restricted to $\mathbb{Z}$ solves the discrete problem.

Traveling Waves of Continuum Problem

$$\dot{v}(\eta,t) = v(\eta+1,t) - 2v(\eta,t) + v(\eta-1,t) - \beta f(v(\eta,t))$$

- Introduce a TW ansatz as before:

Let $x = \eta - \hat{c}t$, $u(x,t) = v(\eta - \hat{c}t, t)$

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Let $x = \eta - \hat{c}t$, $u(x,t) = v(\eta - \hat{c}t, t)$

$$\frac{\partial u(x,t)}{\partial t} - \hat{c}\frac{\partial u(x,t)}{\partial x} = u(x+1,t) - 2u(x,t) + u(x-1,t) - \beta f(u(x,t))$$

- Now have a PDE to solve!
- Have we gained anything?

Correspondence of Solutions

Compare the 2 TW equations

$$
\begin{aligned}
-c\varphi'(\zeta) &= \varphi(\zeta+1) - 2\varphi(\zeta) + \varphi(\zeta-1) - \beta f(\varphi(\zeta)) \\
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- Furthermore, for the Discrete Nagumo Equation...

If a monotone fixed point solution of this PDE exists then it is the unique, asymptotically stable, monotonic solution of our original TW equation.

Algorithm in Action

Our new approach is

- Take an Initial Condition and guess $\hat{c}$.
- Adjust $\hat{c}$ so that $\frac{\partial u}{\partial t} \to 0$.
- Resulting fixed point solution is $\varphi(\zeta)$ and $\hat{c} = c$.

Details

- A finite difference approach is used to evolve the solution.
- For the computations shown here, an explicit method is used with the time step Δt a fixed ratio of the spatial mesh size Δx.

Algorithm in Action

Advances and Delays

- Handled in an analagous fashion to the earlier approach.
- Linearizing the dynamics still yields a PDE.
- Assuming that the fixed point dynamics hold $\dot{u}(x,t) = 0$ then

$$
\begin{aligned}
-\hat{c}\frac{\partial u}{\partial x}(x,t) &= u(x+1,t) - 2u(x,t) + u(x-1,t) \\
&- \beta f'(0)u(x,t)
\end{aligned}
$$

Algorithm in Action

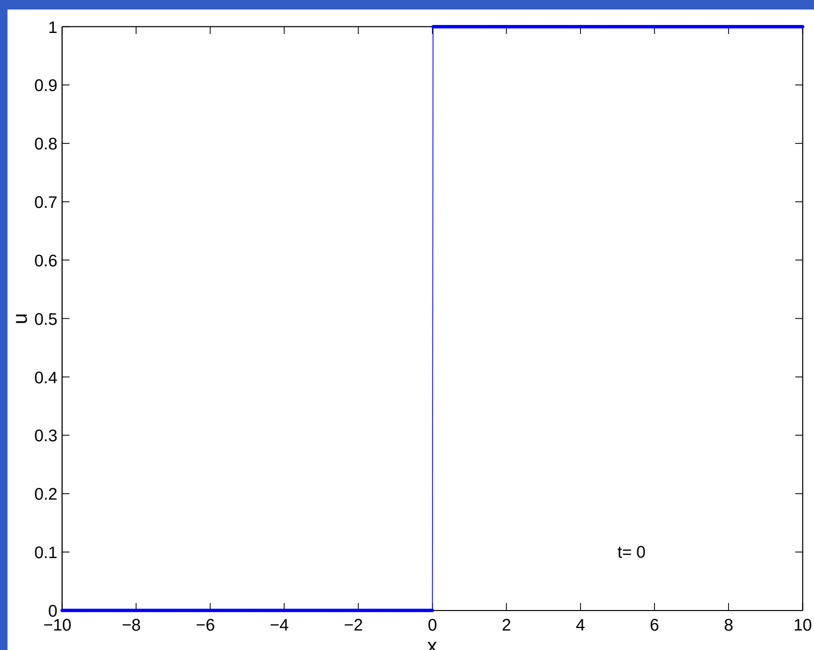
- Fundamental modes can be obtained $e^{x\lambda}$ by solving the characteristic equation.
- Approximation solution to the linearized dynamics is obtained.

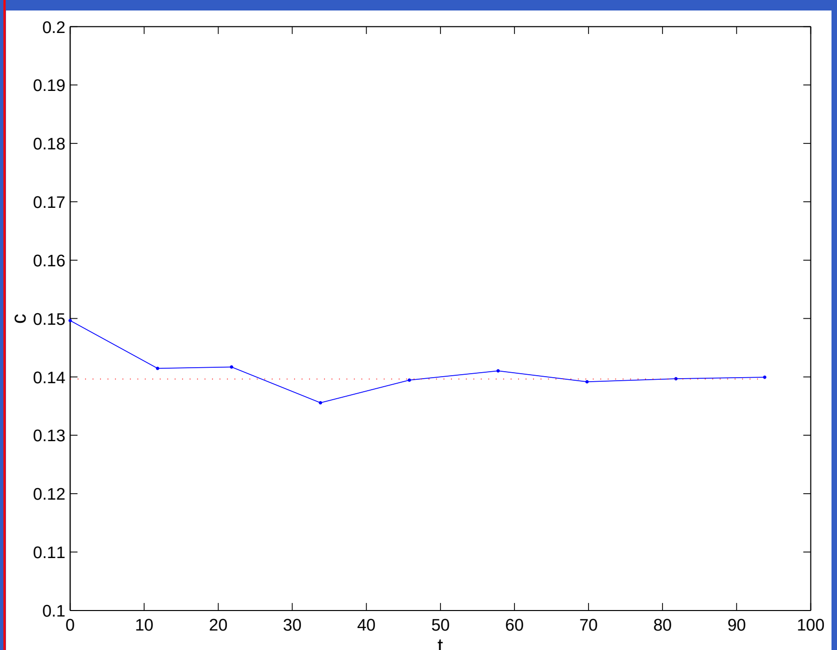
$$u(x,t) \approx \sum_{j=0}^{M} \kappa_j e^{x\lambda_j}$$

- The κ_j are chosen at each time t to satisfy matching conditions

Sample Computations

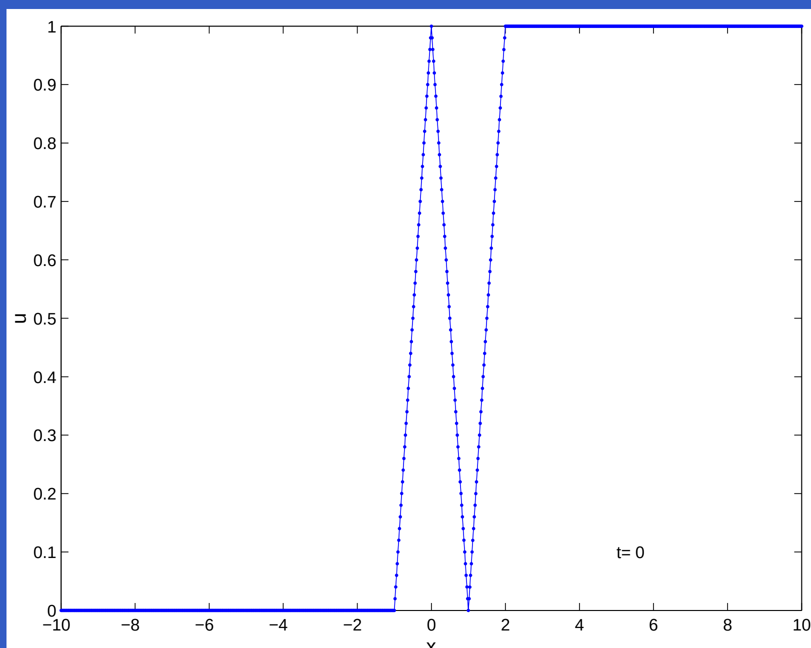
Step Function IC

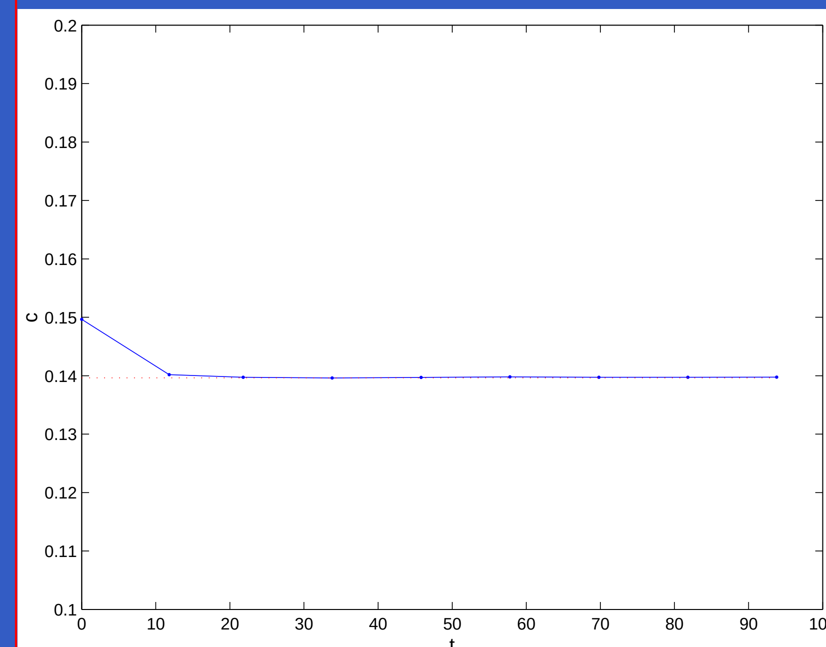




Sample Computations

Non-monotonic

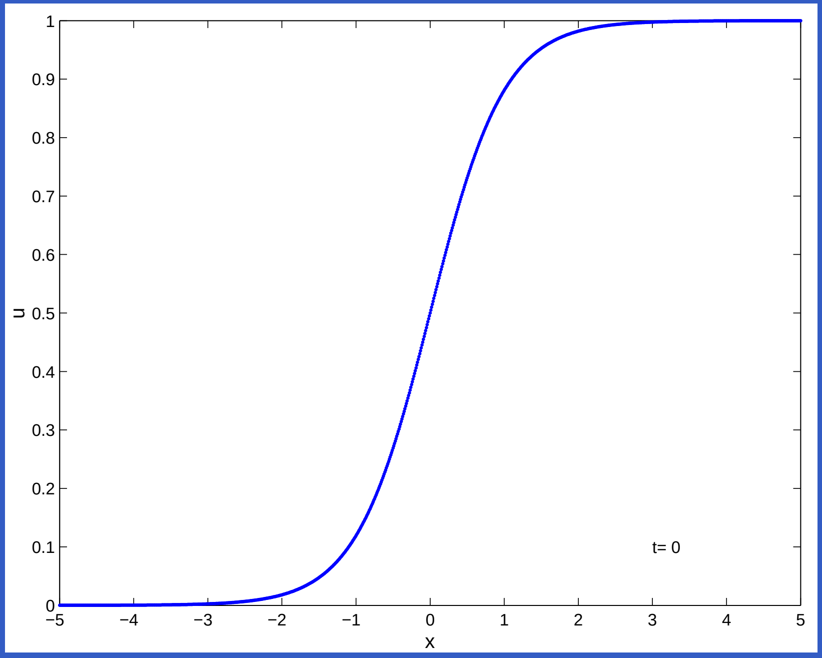


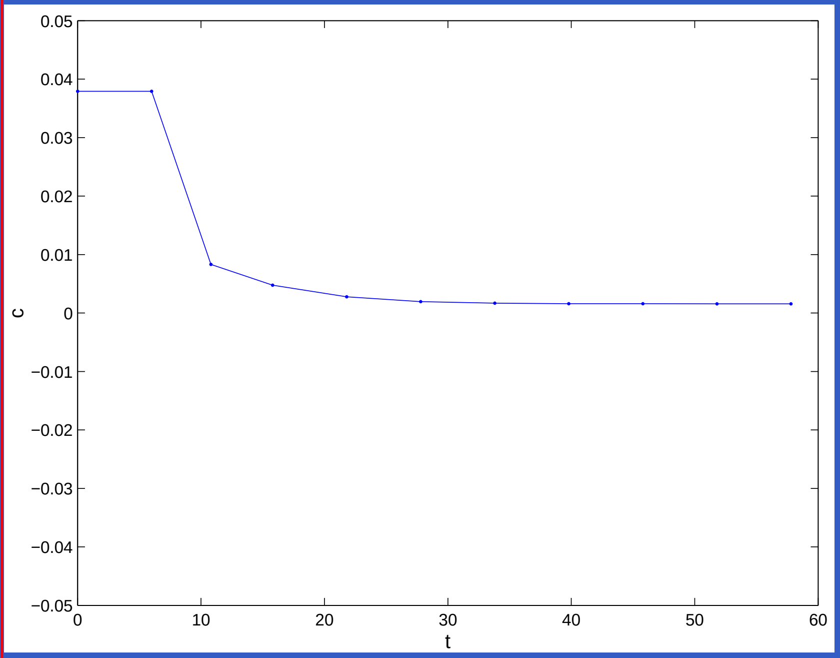


Capture Propagation Failure

- Increasing β and adjusting the de-tuning parameter a so that $c \to 0$

Smooth $\tanh$ IC with $\beta = 8, a = 0.52$

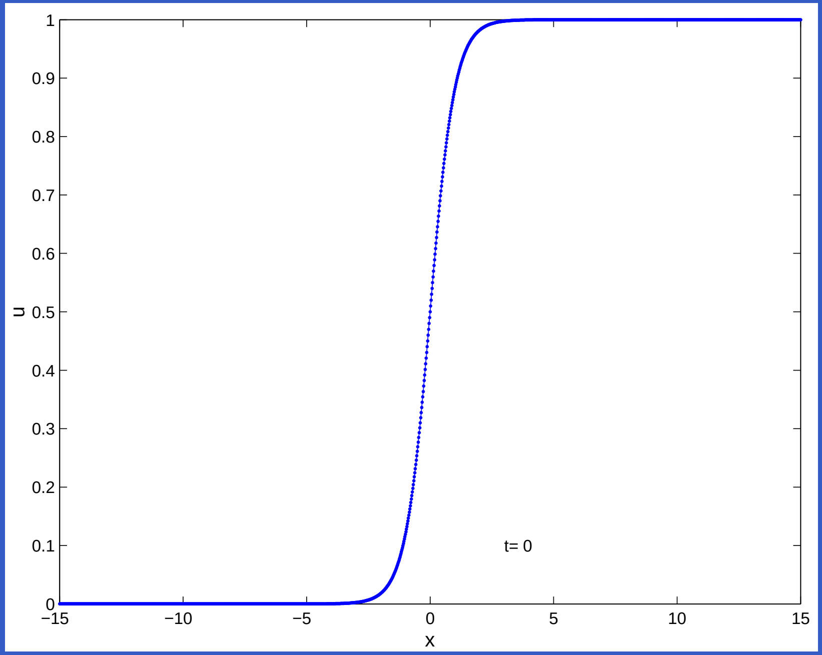


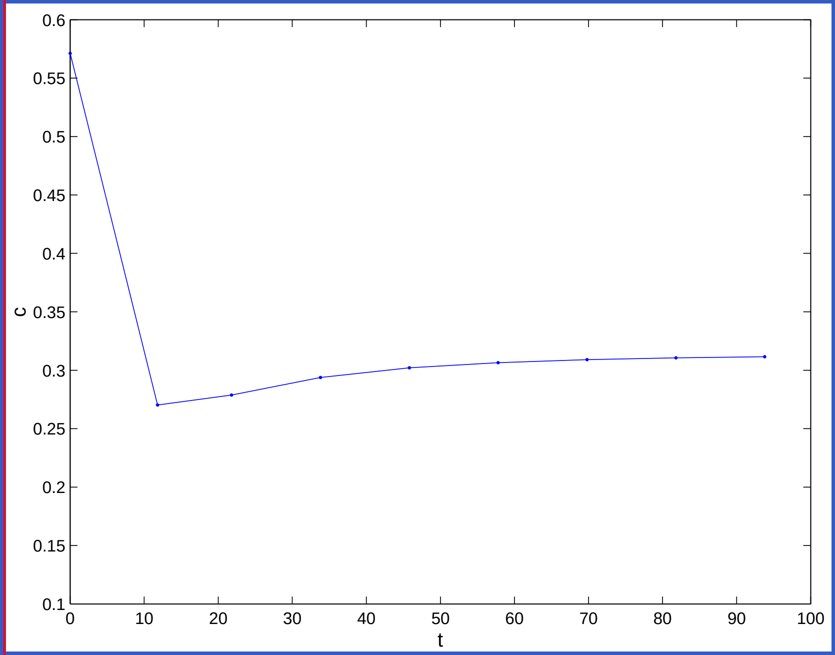


Quintic Nonlinearity

$$f(x) = x(x-b)(x-\frac{1}{2})(x-a)(x-1) \quad a, b \in (0,1)$$

Smooth $\tanh$ IC with $\beta = 10, a = 0.9, b = 3/8$

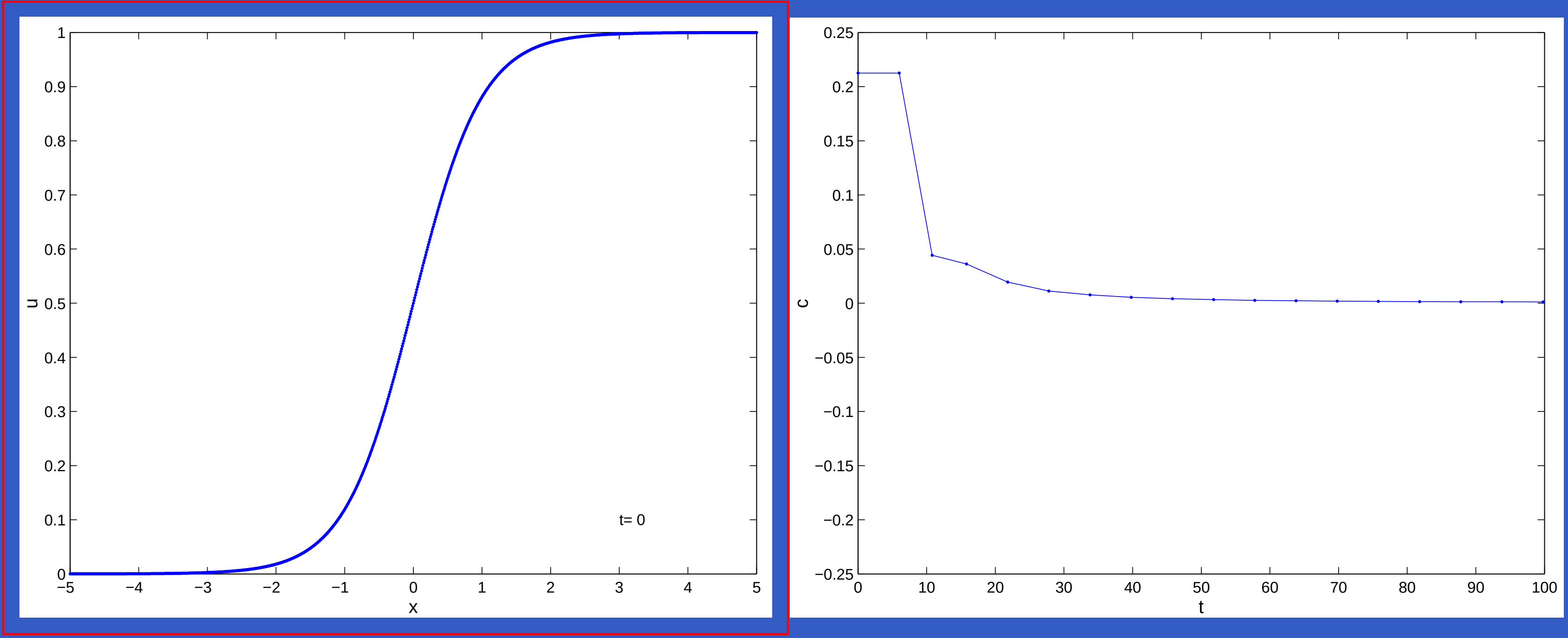




Quintic Nonlinearity

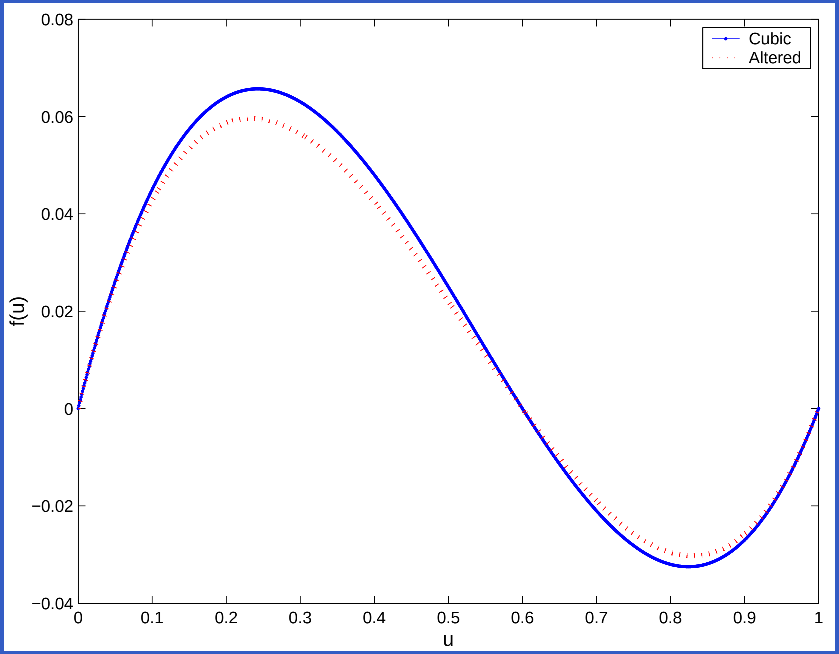
Capture Propagation Failure

Smooth $\tanh$ IC with $\beta = 40, a = 5/8 + 0.1, b = 3/8$



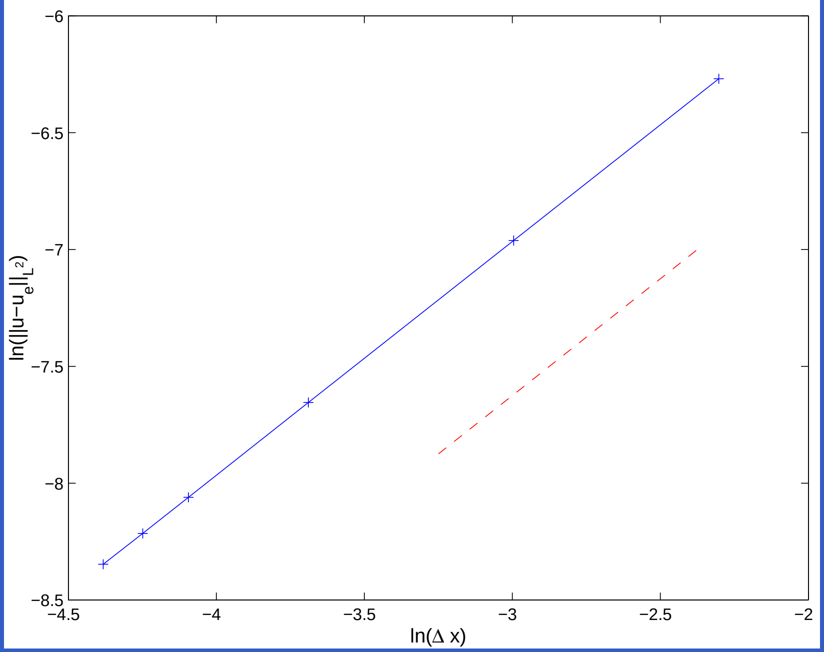
Convergence Results

Using a slightly modified function $f_e(u)$ we have access to an exact solution of our problem.



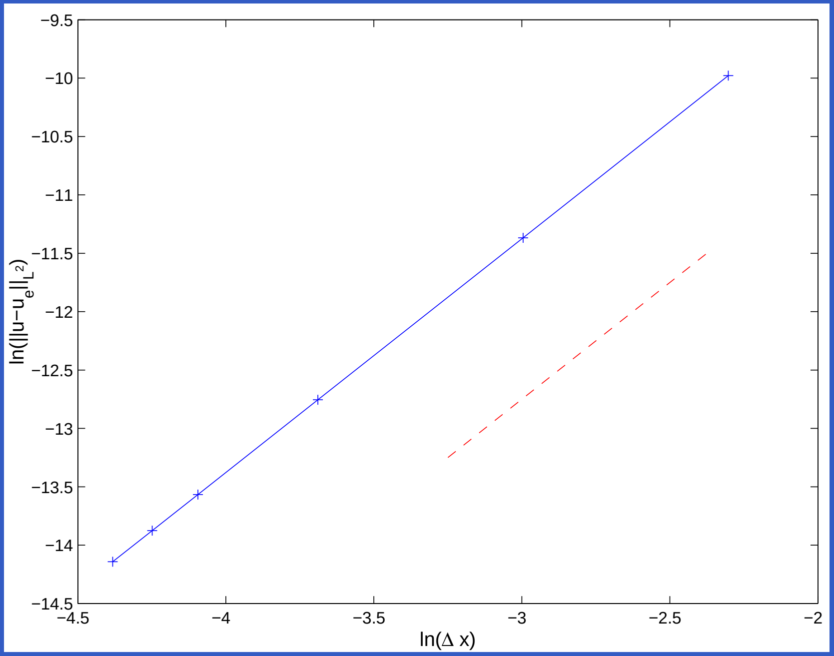
Convergence Results

1^{st} order method



Convergence Results

2^{nd} order method



Conclusions

- The new continuum method allows an **explicit** approach (via Finite Differences) to obtain solutions.
- Our method may be applied to a variety of Lattice Differential Equations.
- Approach is restricted to finding monotonic, stable solutions.

Acknowledgments

- Supervisor: Tony Humphries
- The Applied Math Group at McGill University

Thank you

Any Questions?

Multiple Traveling Fronts

$$f(\varphi) = \sin(4\pi\varphi) + \sin(3\pi\varphi)$$

Smooth $\tanh$ IC