Lattices, Travelling Waves, and Differential Equations with Retarded and Advanced Arguments

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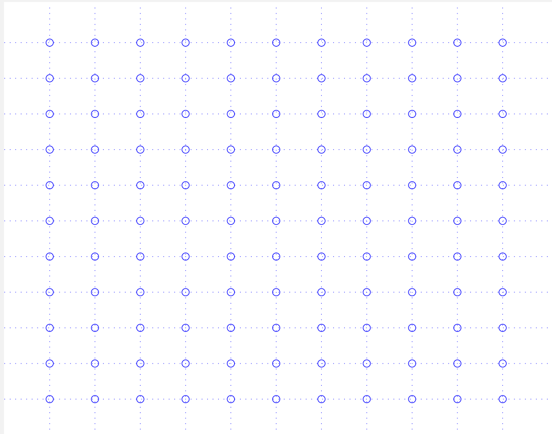
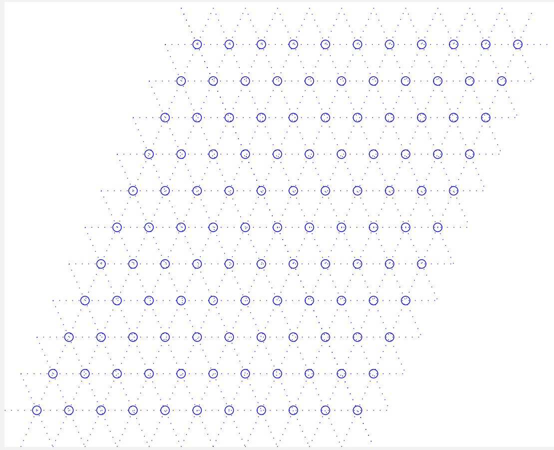
Many lattice differential equations appear to admit standing or travelling wave solutions. In applications (e.g., crystal growth, nerve conduction) it is important to know which occurs. However, this is a difficult question to answer, because travelling waves on lattices are defined by Advanced-Retarded Functional Differential Equations with the propagation failure limit as the wave speed vanishes being singular, whereas standing waves are defined by difference equations.

A typical LDE has the form

$$\dot{u}_i = g_i(\{u_j\}_{j\in\Lambda}), \quad i \in \Lambda.$$

$\Lambda \subset \mathbb{R}^n$ is a lattice; a discrete subset of $\mathbb{R}^n$, finite or infinite number of points, regular spatial structure (eg $\mathbb{Z}^n$)





- $u_i(t)$ for each $i \in \Lambda$ may be scalar or vector
- Continuous in time, discrete in space
- Today will restrict attention to 1D lattices for simplicity

Travelling Pulse Model

Discrete Fitzhugh-Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - f(u_i) - v_i$$

$$\dot{v}_i = b(u_i - rv_i)$$

Models a Myelinated Nerve fibre

- lattice points i represent nodes of Ranvier; gaps in myelin sheath where nerve may be excited
- u_i represents transmembrane potential at node i
- v_i is a recovery variable (potassium current)

Normal myelinated Fibre [Experiment] [HUGH BOSTOCK, UNIVERSITY OF LONDON].

Discrete Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - \beta f(u_i) \qquad \beta > 0,$$

Models leading edge behaviour of pulse. Two examples:

Discrete Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - \beta f(u_i) \qquad \beta > 0,$$

Models leading edge behaviour of pulse. Two examples:

1. Cubic nonlinearity

$$f(u) = u(u-a)(u-1)$$

[Travelling Wave Movie] [Standing Wave Movie]

2. McKean's caricature of cubic

$$f(u) = \begin{cases} u-1, & u > a, \\ [u-1, u], & u = a, \\ u, & u < a. \end{cases}$$

[Travelling Wave Movie] [Standing Wave Movie]

Nagumo PDE Model

- Consider the PDE
$$u_t = u_{xx} - f(u), \quad x \in \mathbb{R},$$
with cubic nonlinearity $f(u) = u(u-a)(u-1)$ models leading edge behaviour of pulse in the squid giant axon.
- Homogeneous steady states satisfy $f(u) = 0$ implies $u = 0$, $u = a$ or $u = 1$. Natural to look for solutions $\lim_{x \to -\infty} u(x,t) = 0$, $\lim_{x \to +\infty} u(x,t) = 1$.

Travelling Wave ansatz:

- Let $u(x,t) = \varphi(x-ct) = \varphi(\xi)$, where c is unknown wave speed

$$u_t = u_{xx} - f(u)$$

Travelling Wave Ansatz
PDE to ODE reduction

Travelling Wave ansatz:

- Let $u(x,t) = \varphi(x - ct) = \varphi(\xi)$, where c is unknown wave speed

$$u_t = u_{xx} - f(u) \Rightarrow -c\varphi'(\xi) = \varphi''(\xi) - f(\varphi(\xi)),$$

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- TW ansatz reduces PDE to ODE
- ξ is time-like variable
- Boundary conditions
  $\varphi(-\infty) = 0$, $\varphi(\infty) = 1$.
- Solutions not unique (translational invariance)

Functional Differential Equation Reduction

Travelling Waves for Discrete Nagumo Equation

$$\dot{u}_i = (u_{i+1} - 2u_i + u_{i-1}) - \beta f(u_i)$$

- Travelling Wave ansatz $u_i(t) = \varphi(i - ct) = \varphi(\xi)$ gives

$$-c\varphi'(\xi) = \varphi(\xi + 1) - 2\varphi(\xi) + \varphi(\xi - 1) - \beta f(\varphi(\xi))$$

Functional Differential Equation Reduction
Travelling Waves for Discrete Nagumo Equation

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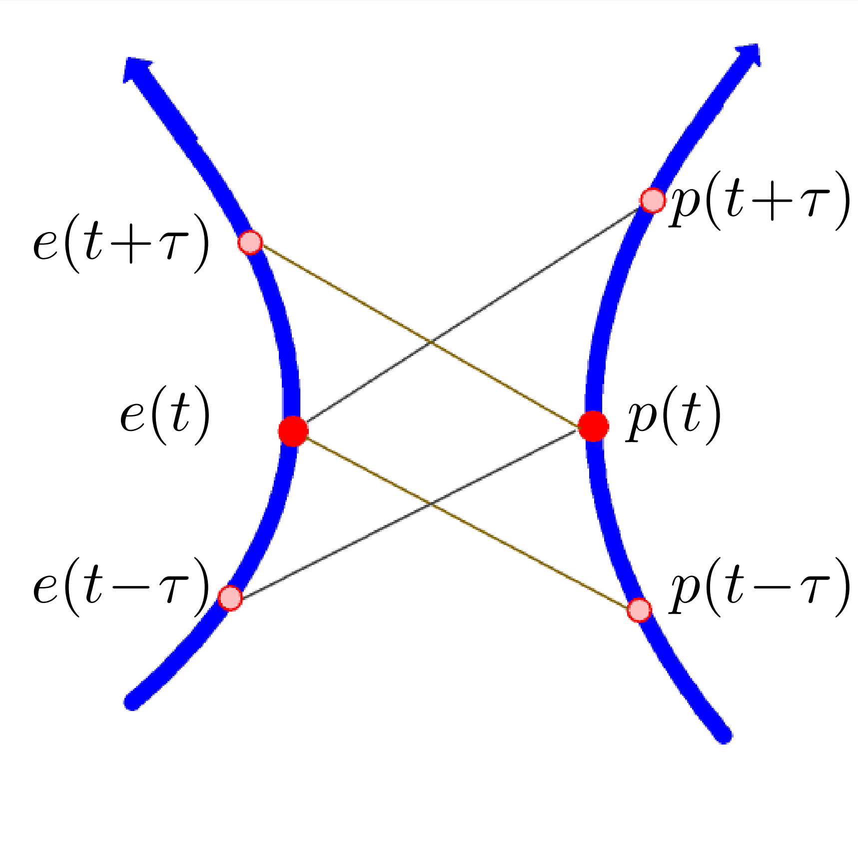
- $i \in \mathbb{Z}$ but $\xi = i - ct \in \mathbb{R}$ is time-like and $\varphi : \mathbb{R} \to \mathbb{R}$.
- $\varphi(\xi - 1) =$ delay, $\varphi(\xi + 1) =$ advance.
- Both nonlinearities have three constant solutions $\varphi \equiv 0$, a and 1. Seek solutions with $\varphi(-\infty) = 0$, $\varphi(\infty) = 1$.
- Propagation Failure $c \to 0$ is singular limit
- TW ansatz “reduces” LDE to a mixed-type FDE !!

Another Example Of
Mixed Type Functional Differential Equations



$$m_e\ddot{e} = F_{ep} = \frac{1}{2}(F_{ep}^{+} + F_{ep}^{-})$$

$$m_p\ddot{p} = F_{pe} = \frac{1}{2}(F_{pe}^{+} + F_{pe}^{-})$$

$$F_{ep}^{\pm}(t) = -K\frac{p(t\pm\tau) - e(t)}{|p(t\pm\tau) - e(t)|^3}$$

$$F_{pe}^{\pm}(t) = -K\frac{e(t\pm\tau) - p(t)}{|e(t\pm\tau) - p(t)|^3}$$

$$|p(t\pm\tau) - e(t)| = c\tau$$

$$|e(t\pm\tau) - p(t)| = c\tau$$

- [Wheeler Feynman 1945&1949],[Schild 1963],[Many Others...]

Nonlinear FDE BVP

Existence and Uniqueness

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta\varphi(\xi)(\varphi(\xi)-a)(\varphi(\xi)-1)$$

$$\varphi(-\infty) = 0, \quad \varphi(\infty) = 1.$$

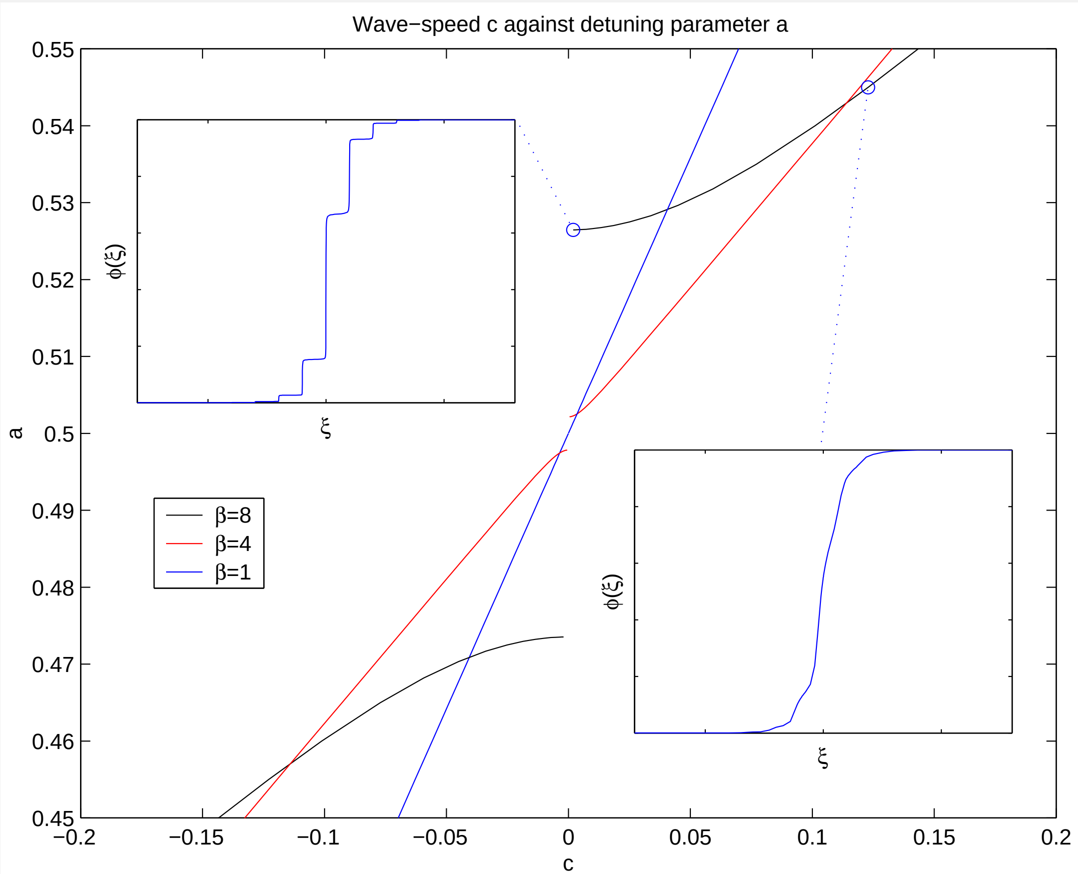
- [ZINNER 1991]: Uniqueness and Stability of Monotonic TWs
- [ZINNER 1992]: Existence of Monotonic TWs for β suff small.
- Zinner's theory covers larger class of f. More recent extensions include, in particular [MALLET-PARET 1999A],[MALLET-PARET 1999B].
- When is wave travelling or standing?
- Question has practical relevance to problem of waveblock in for example MS where signals fail to propagate along a demyelinated nerve.
- Solve TW equations numerically using a mixed-type DDE collocation code written for the purpose [ABELL ET AL 2005], (built on colmod [CASH ET AL 1995]).

Nonlinear Nagumo Equation

a-c curves

$$\dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta u_i(u_i - a)(u_i - 1), \quad \beta > 0$$

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta\varphi(\xi)(\varphi(\xi) - a)(\varphi(\xi) - 1),$$



β small $\Longrightarrow$
$c = 0 \iff a = 1/2$???

β large $\Longrightarrow$
$c = 0$ for growing range of a:
$=$ Propagation Failure

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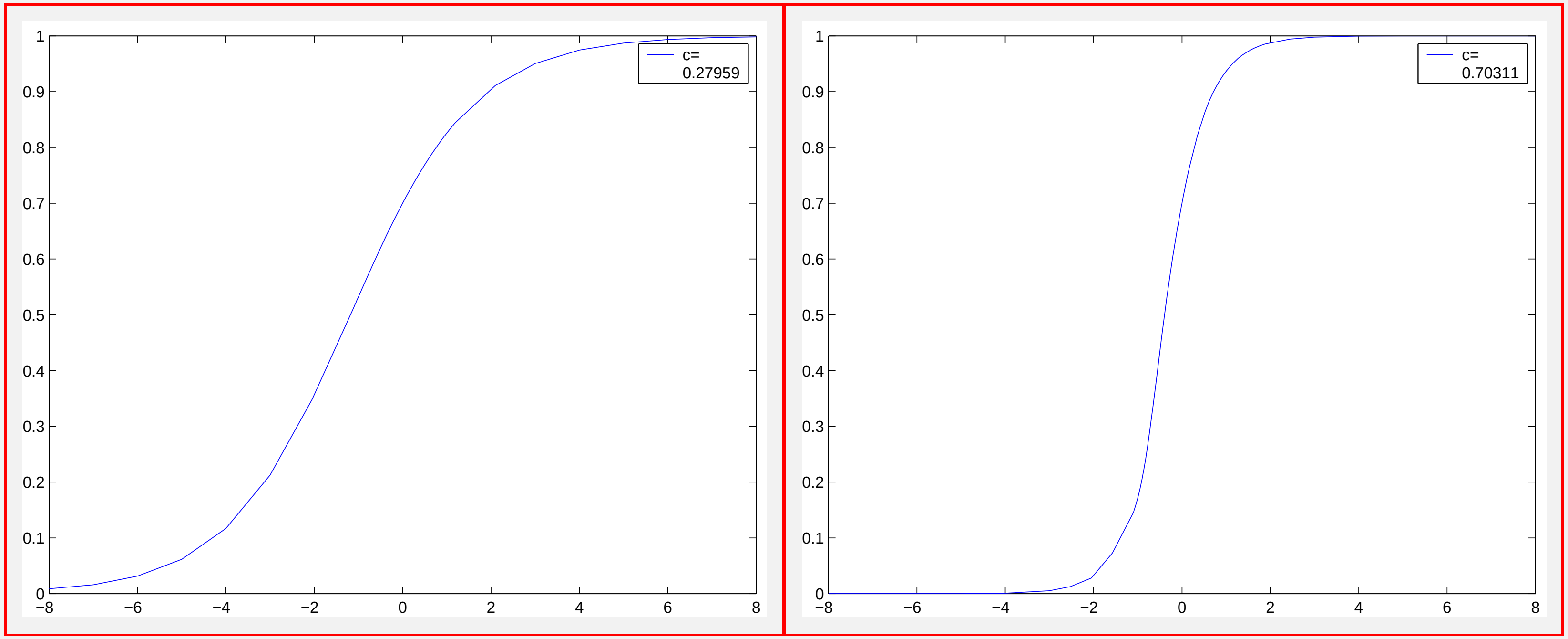
Nonlinear Nagumo Equation

Evolution of Wave Profile for $\beta = 1$ and $\beta = 8$.

$$-c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta\varphi(\xi)(\varphi(\xi)-a)(\varphi(\xi)-1)$$



- Consider evolution of wave profile as $c \to 0$

Nonlinear Nagumo Equation

Evolution of Wave Profile for $\beta = 1$ and $\beta = 8$.

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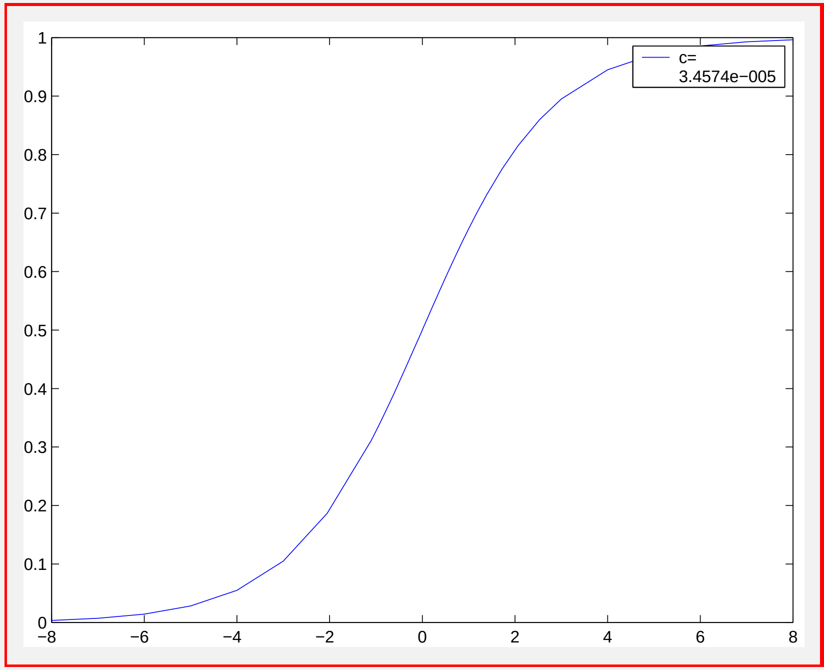
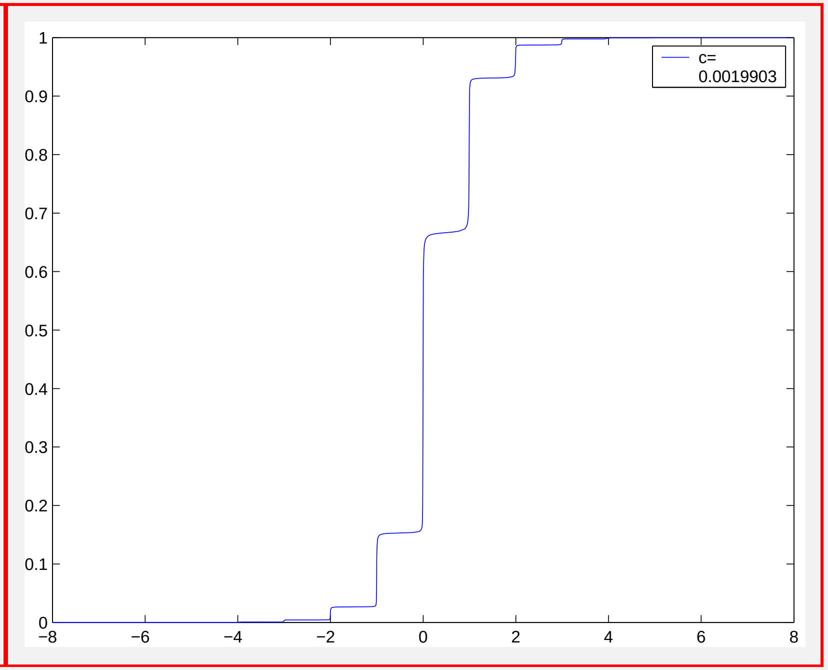




- Consider evolution of wave profile as $c \to 0$
- Step profile explains this
- TW equation becomes a difference equation

Propagation Failure & Standing Waves

$c = 0$: A difference Equation

$$0 = -c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta f(\varphi(\xi))$$

$$0 = \dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta f(u_i)$$

McKean's caricature of cubic

$$f(\varphi) = \begin{cases} \beta(\varphi - 1), & \varphi > a, \\ \beta[\varphi - 1, \varphi], & \varphi = a, \\ \beta\varphi, & \varphi < a. \end{cases} = \varphi - H(\varphi - a),$$

where H is heaviside function.

Propagation Failure & Standing Waves

$c = 0$: A difference Equation

$$0 = u_{i+1} - 2u_i + u_{i-1} - \begin{cases} \beta(u_i - 1), & u_i > a, \\ \beta u_i, & u_i < a. \end{cases}$$

Consider monotonic solution s.t. $\begin{cases} u_i < a, & i < 0, \\ u_i > a, & i \geqslant 0. \end{cases}$

$$x^2 - (2+\beta)x + 1 = 0 \quad \Longrightarrow \quad x = 1 + \frac{\beta}{2} + \frac{1}{2}\sqrt{4\beta + \beta^2}$$

Solution:

$$u_i = \frac{x^{i+1}}{x+1}, \quad i \leq 0, \qquad u_i = 1 - \frac{1}{x^i(x+1)}, \quad i \geqslant -1$$

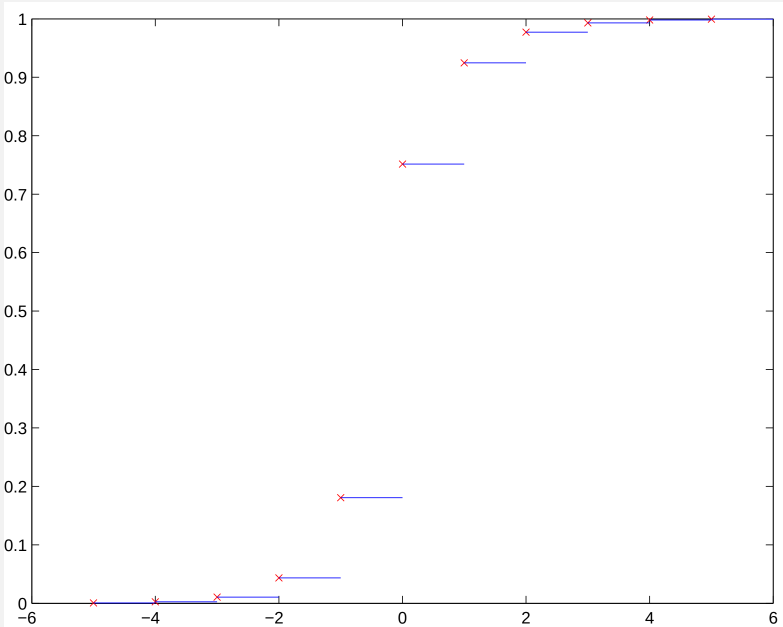
valid for

$$u_{-1} = \frac{1}{1+x} < a < 1 - \frac{1}{1+x} = u_0$$

$c = 0$: A difference Equation

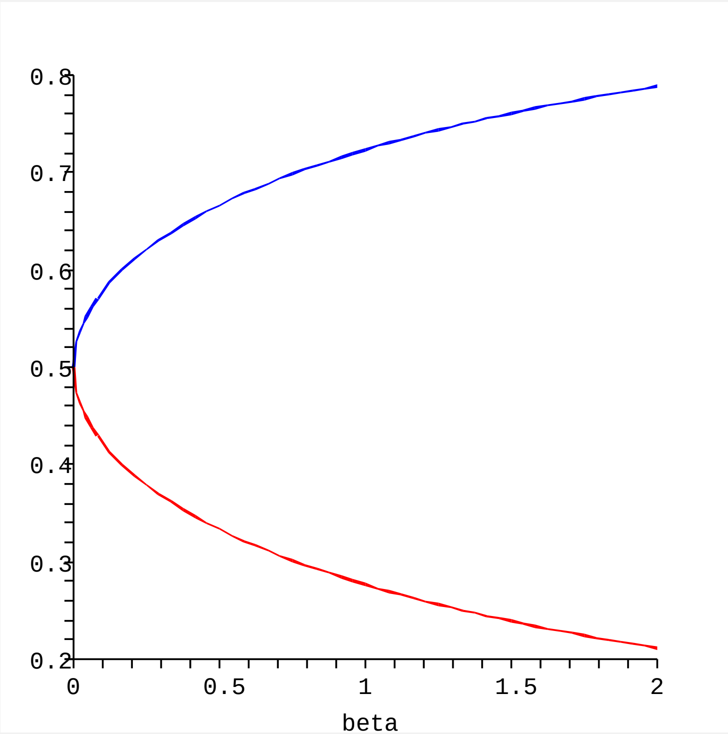
$$0 = -c\varphi'(\xi) = \varphi(\xi+1) - 2\varphi(\xi) + \varphi(\xi-1) - \beta f(\varphi(\xi))$$

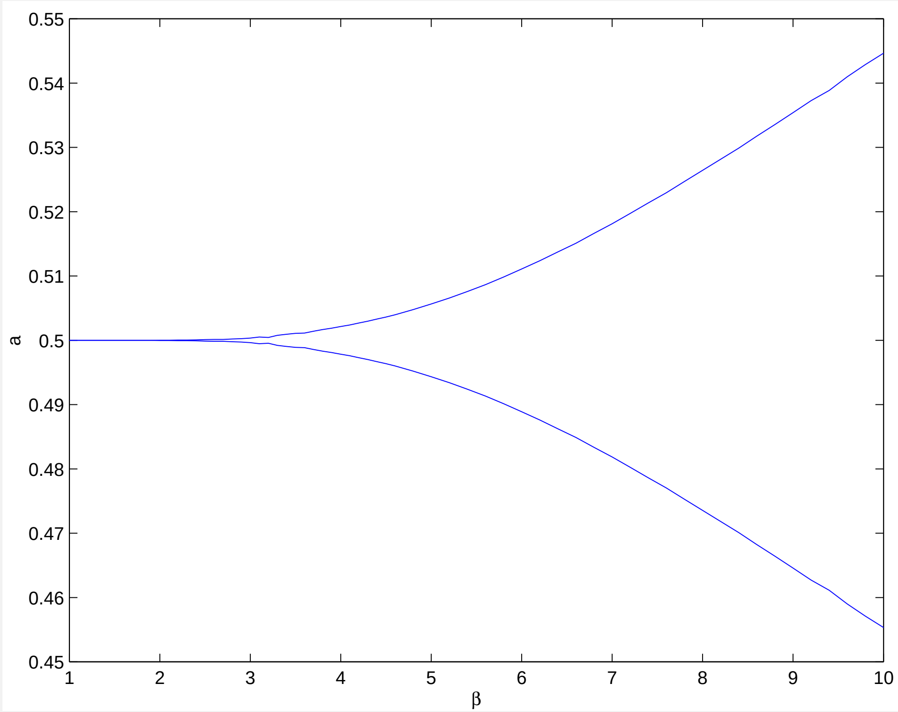
$$0 = \dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta f(u_i)$$



Propagation Failure

McKean and Cubic Nonlinearities





- McKean: $\forall\beta > 0\ \exists\varepsilon > 0 : c = 0$ for $a \in [1/2 - \varepsilon, 1/2 + \varepsilon]$
- Cubic: $\beta \gg 0$ = ditto
- Cubic: $\beta \approx 0$: $\varepsilon = 0$ or $\mathcal{O}(e^{-1/\beta})$???
- Why not always steps in wave profile as $c \to 0$???

Discrete Nagumo Standing Waves As Hamiltonian Discretizations

$$0 = \dot{u}_i = u_{i+1} - 2u_i + u_{i-1} - \beta f(u_i)$$

Let $h = \sqrt{\beta}$ and $v_{j+1} = (u_{j+1} - u_j)/h$. Then

$$u_{j+1} = u_j + hv_{j+1}, \qquad v_{j+1} = v_j + hf(u_j)$$

Standing wave of LDE is a heteroclinic connection of this mapping between $(0, 0)$ and $(1, 0)$ in (u, v)-plane.

Discrete Nagumo Standing Waves As Hamiltonian Discretizations

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Which is symplectic Euler applied to the Hamiltonian system

$$\dot{v} = -H_u(u, v), \qquad \dot{u} = v = H_v(u, v)$$

where

$$H(u, v) = \frac{v^2}{2} - W(u), \quad W'(u) = f(u)$$

For cubic f, W is a quartic double-well potential

The standing wave is a heteroclinic connection between $(u, v) = (0, 0)$ and $(1, 0)$. But we can do better...

Stormer-Verlet Discretization

$$u_{i+1} - 2u_i + u_{i-1} = h^2 f(u_i)$$

Let $v_j = \frac{1}{2h}(u_{j+1} - u_{j-1})$ then

$$u_{j+1} = u_j + hv_j + \frac{1}{2}h^2 f(u_j), \qquad v_{j+1} = v_j + \frac{h}{2}f(u_j) + \frac{h}{2}f(u_{j+1}),$$

or

$$\begin{aligned} v_{j+\frac{1}{2}} &= v_j + \frac{h}{2}f(u_j), \\ u_{j+1} &= u_j + hv_{j+\frac{1}{2}}, \\ v_{j+1} &= v_{j+\frac{1}{2}} + \frac{h}{2}f(u_{j+1}). \end{aligned}$$

Which is Stormer-Verlet method for $\dot{u} = v$, $\dot{v} = f(u)$.

Stormer-Verlet Discretization

Standing wave of LDE is a heteroclinic connection of Stormer-Verlet method

$$u_{j+1} = u_j + hv_j + \frac{1}{2}h^2 f(u_j), \qquad v_{j+1} = v_j + \frac{h}{2}f(u_j) + \frac{h}{2}f(u_{j+1}),$$

between $(0, 0)$ and $(1, 0)$ in (u, v)-plane, again applied to the Hamiltonian system

$$\dot{v} = -H_u(u, v), \qquad \dot{u} = v = H_v(u, v)$$

where

$$H(u, v) = \frac{v^2}{2} - W(u), \quad W'(u) = f(u)$$

Stormer-Verlet as well as being symplectic and explicit is also second order and symmetric.

Standing Wave

Recall TW for $u_t = u_{xx} - f(u)$ given by $u(x,t) = \varphi(x - ct) = \varphi(\xi)$ satisfying ODE

$$-c\varphi'(\xi) = \varphi''(\xi) - f(\varphi(\xi)).$$

Note standing wave $c = 0$ is not a singular limit

$$\varphi''(\xi) = f(\varphi(\xi)),$$

or

$$\varphi'(\xi) = \psi(\xi), \qquad \psi'(\xi) = f(\varphi(\xi)),$$

which is Hamiltonian System

$$\psi' = -H_\varphi(\varphi, \psi), \qquad \varphi' = \psi = H_\psi(\varphi, \psi)$$

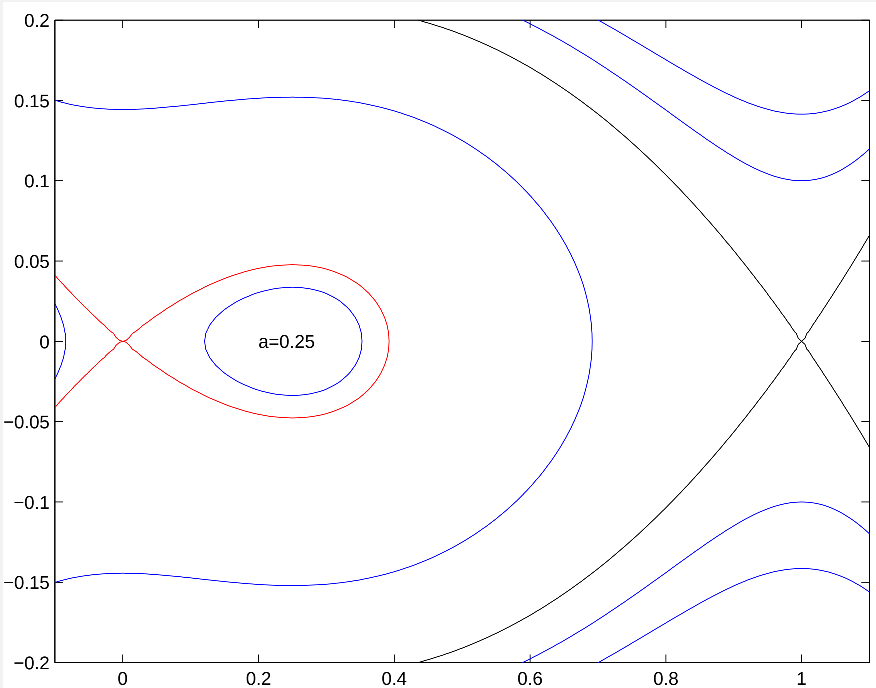
where

$$H(\varphi, \psi) = \frac{\psi^2}{2} - W(\varphi), \quad W_\varphi(\varphi) = f(\varphi)$$

- Thus for all $\beta > 0$ discrete Nagumo Standing Wave problem corresponds to a Stormer-Verlet discretization with $h = \sqrt{\beta}$ of continuous Nagumo Standing Wave Problem

$$\psi' = -H_\varphi(\varphi, \psi, a), \qquad \varphi' = H_\psi(\varphi, \psi, a)$$

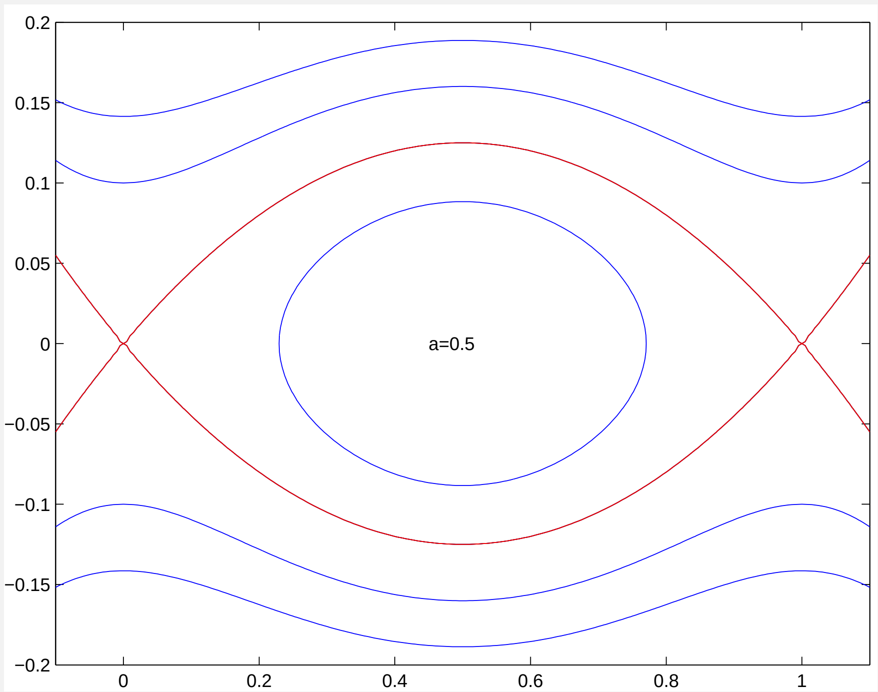
$$H(\varphi, \psi, a) = \frac{\psi^2}{2} - W(\varphi, a), \quad W_\varphi(\varphi, a) = f(\varphi, a)$$



Nagumo PDE
Standing Wave

$$\psi' = -H_\varphi(\varphi, \psi, a), \qquad \varphi' = H_\psi(\varphi, \psi, a)$$

$$H(\varphi, \psi, a) = \frac{\psi^2}{2} - W(\varphi, a), \quad W_\varphi(\varphi, a) = f(\varphi, a)$$



Standing Wave

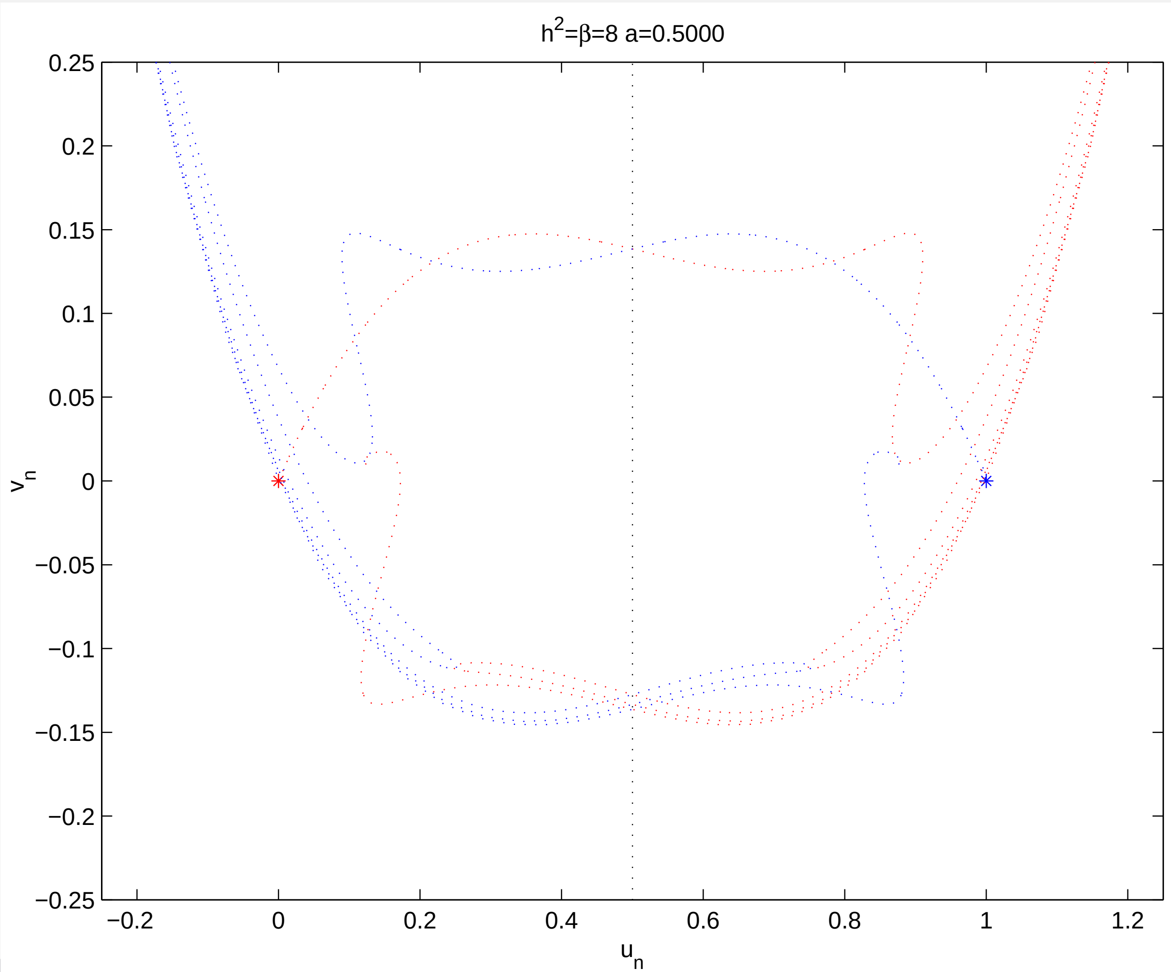
$$\psi' = -H_\varphi(\varphi, \psi, a), \qquad \varphi' = H_\psi(\varphi, \psi, a)$$

$$H(\varphi, \psi, a) = \frac{\psi^2}{2} - W(\varphi, a), \quad W_\varphi(\varphi, a) = f(\varphi, a)$$

- For Nagumo PDE $c = 0$ iff $a = 1/2$
- Then unstable manifold of $(0, 0)$ and stable manifold of $(1, 0)$ intersect tangentially

Discrete Nagumo Standing Wave

$$u_{j+1} = u_j + hv_j + \frac{1}{2}h^2 f(u_j), \qquad v_{j+1} = v_j + \frac{h}{2}f(u_j) + \frac{h}{2}f(u_{j+1}),$$



$\beta = 8$
$h = \sqrt{\beta}$
$a = 0.5$

Manifolds intersect transversally

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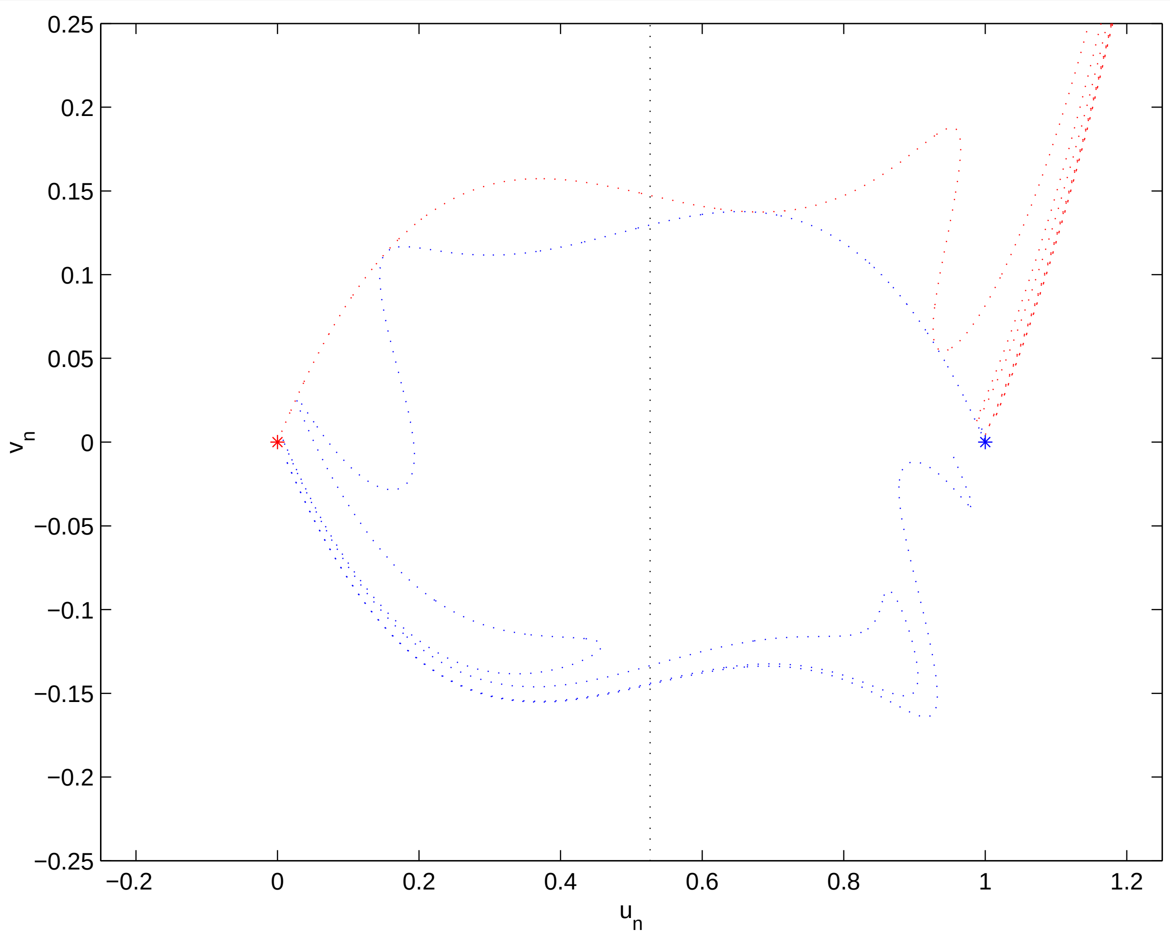
Discrete Nagumo Standing Wave

$$u_{j+1} = u_j + hv_j + \frac{1}{2}h^2 f(u_j), \qquad v_{j+1} = v_j + \frac{h}{2}f(u_j) + \frac{h}{2}f(u_{j+1}),$$



$\beta = 8$
$h = \sqrt{\beta}$
$a = 0.5265$

Connections persist for $a \neq 1/2$
$\Longrightarrow$ Interval of propagation failure

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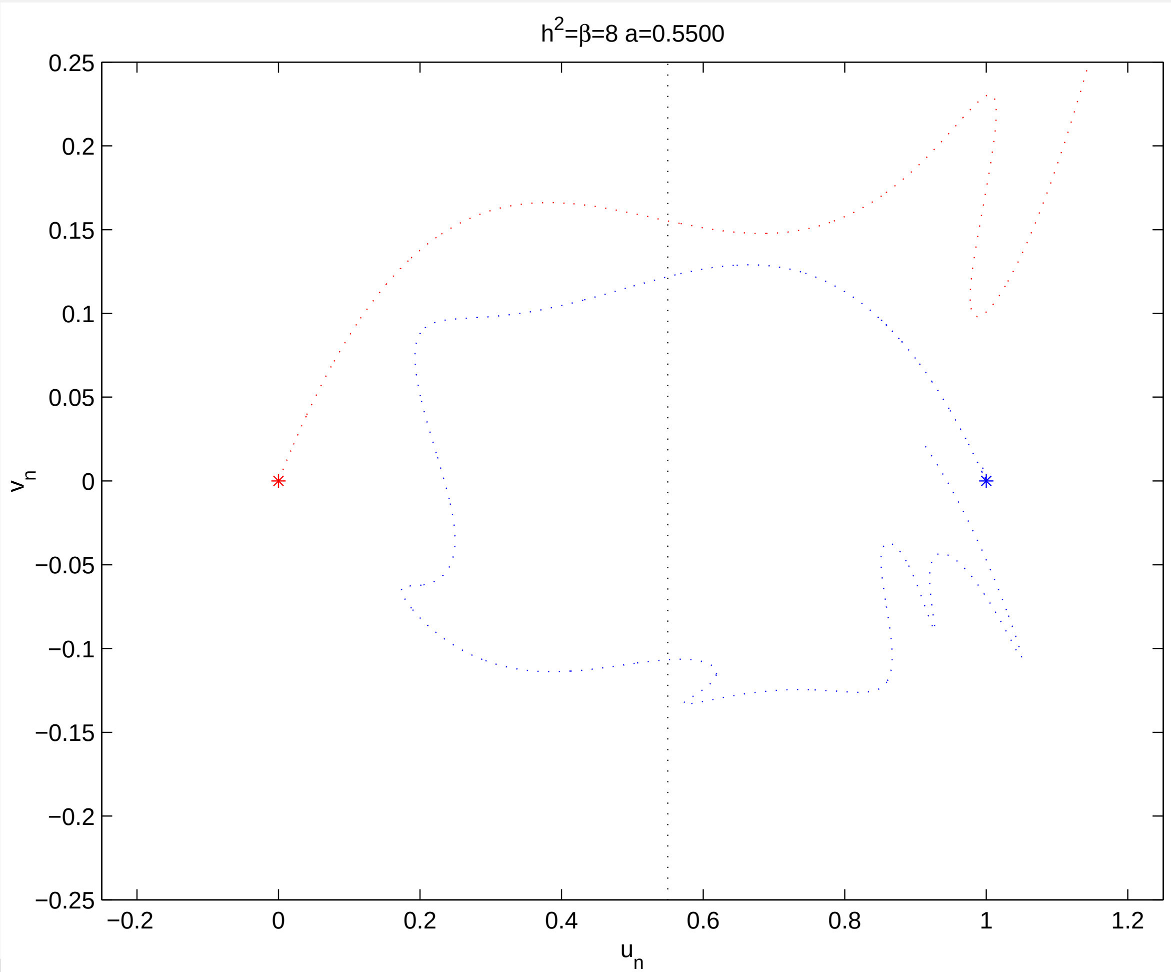
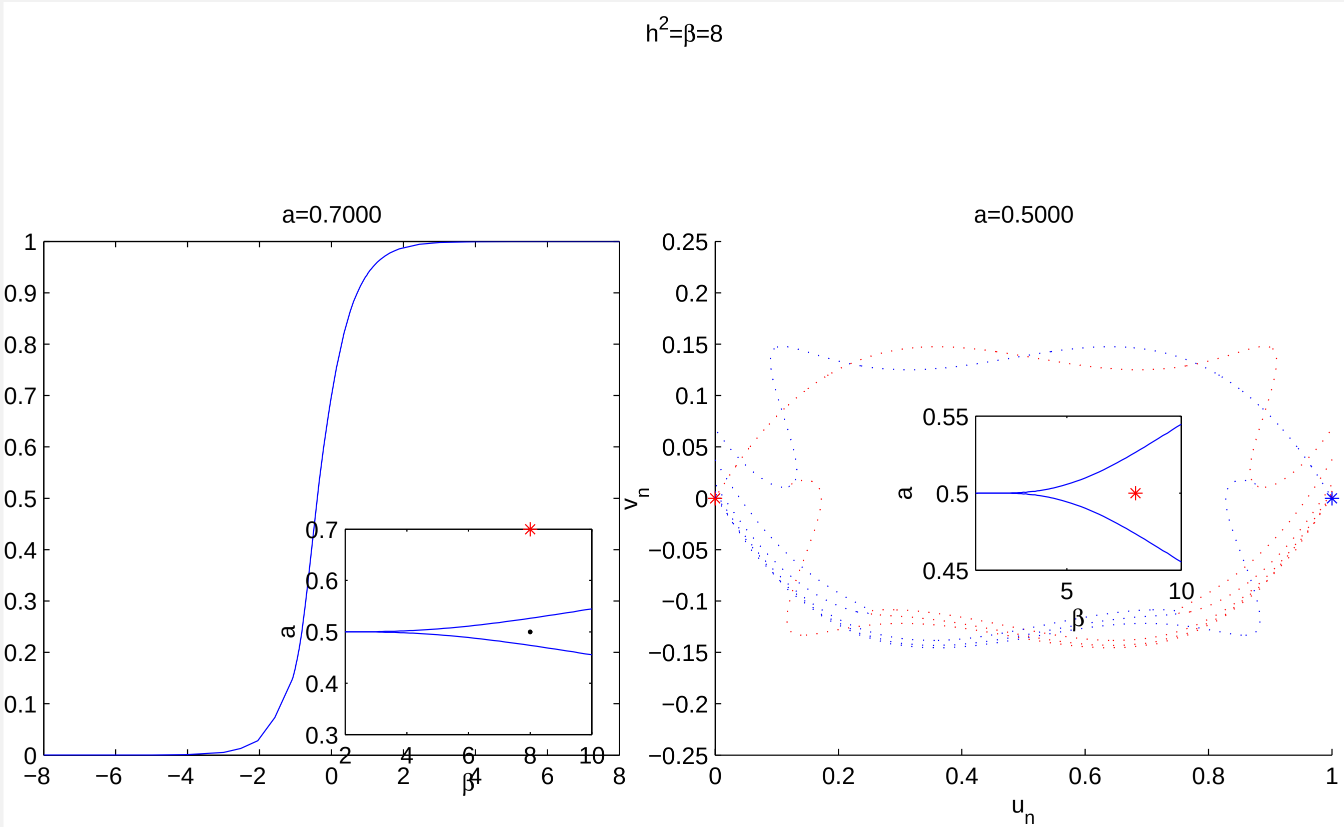
$$u_{j+1} = u_j + h v_j + \frac{1}{2} h^2 f(u_j), \qquad v_{j+1} = v_j + \frac{h}{2} f(u_j) + \frac{h}{2} f(u_{j+1}),$$



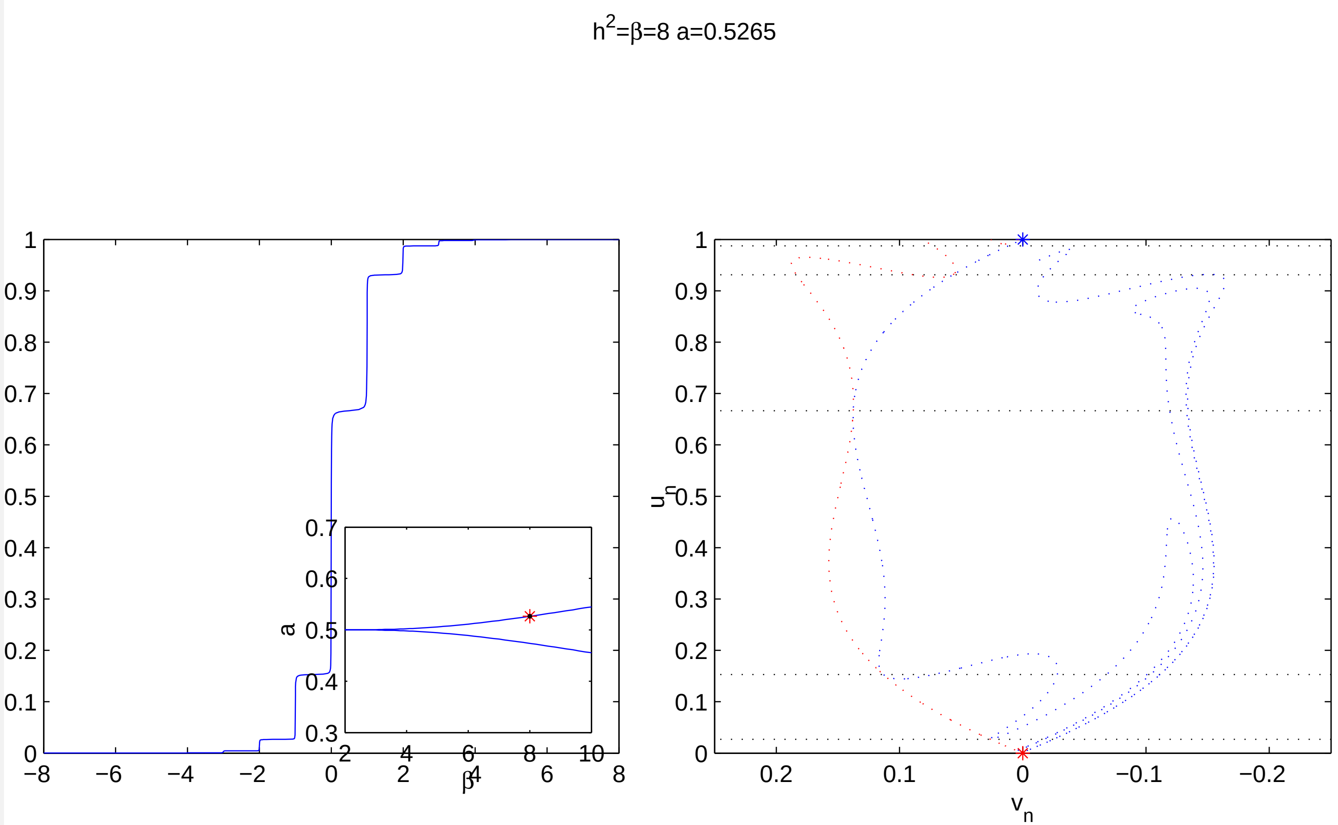
$\beta = 8$
$h = \sqrt{\beta}$
$a = 0.55$

Discrete Nagumo

Standing/Travelling Wave Boundary



Standing/Travelling Wave Boundary



Discrete Nagumo Standing Wave

- Expect heteroclinic connection for discretization for nearby parameter value [Beyn,1990],[Doedel&Friedman,1990]
- In general stable-unstable manifold intersection for discrete map generally transversal; so heteroclinic orbit will persist over (exponentially) small parameter range [Fiedler & Scheurle, 1996]
- Manifolds 'touch' at boundary of interval of propagation failure
- Generically should expect propagation failure for cubic for β small
- Missing steps as $c \to 0$ in numerical computations for $\beta \approx 0$, are result of modified equations argument. Computed wave form is spurious, but is "exact" solution for perturbed continuous problem, and has exponentially small residual for stated problem

Conclusions

- Advanced-Retarded FDE theory is incomplete
- Good numerics are needed to inform analysis
- Analysis is needed to inform numerics
- Can exploit discrete dynamical systems & symplectic method theory
- And difference equations are just numerical methods in disguise!