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Mosaic solutions and entropy for discrete coupled phase-transition equations

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Abstract

We consider arrays of coupled scalar differential equations organized on a spatial lattice. One component of this system is analogous to the Allen–Cahn partial differential equation which on the integer lattice has interactions of nearest neighbor type, and the other to the Cahn–Hilliard partial differential equation which has interactions of nearest and next nearest neighbor type. Our coupling functions are forms of the so-called double obstacle nonlinearity. The interaction strengths of both equations are not restricted in magnitude or in sign and need not be near a continuum limit. We prove existence and uniqueness results for the initial value problem and consider the existence and stability of a class of equilibrium solutions called mosaic solutions. These equilibrium solutions take only the values $+1$, -1, and 0 at each lattice point. Using the notion of a weakly forward invariant set we provide criteria for weak Lyapunov and weak asymptotic stability. Rigorous results are then obtained for the spatial entropy of these stable mosaic solutions and it is shown that the existence and stability results obtained on the integer lattice can be used to obtain similar results on an arbitrary lattice. Numerical results are presented that illustrate the importance of the analytical results. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In models of phase transitions of polymers in chemical engineering and of alloys in metallurgical engineering one often obtains models that are similar in form to Allen–Cahn equations [3] (where mass is not preserved) and/or Cahn–Hilliard equations [6] (where mass is typically preserved). Multi-component versions of both models have been studied. Our interest is in multi-component models where there is one conserved order parameter, and

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one unconserved order parameter. Such models are well-known in physics and are known as "model C" in the terminology of Hohenberg and Halperin [15].

Cahn and Novick-Cohen [8] derive such a model which results in an Allen–Cahn equation coupled to a Cahn–Hilliard equation through the nonlinear terms. The Allen–Cahn/Cahn–Hilliard system is used to describe simultaneous phase separation and order–disorder transition in a BCC Fe–Al binary alloy. Dorgan [12,13] has used a similar model in studying liquid crystals. More recent work on coupled Allen–Cahn/Cahn–Hilliard equations includes problems with degenerate mobility [11] and finite element numerical solutions [4]. The Allen–Cahn/Cahn–Hilliard system contains both the Allen–Cahn and Cahn–Hilliard equations which often serve as diffuse interface models for limiting sharp interface motion. The Allen–Cahn equation is a diffuse interface model for antiphase grain boundary motion, while the Cahn–Hilliard equation describes phase separation with mass conservation. For certain choices of the concentration or order parameter the system of equations reduces to a single equation.

We study microscopic versions of model C, in particular an Allen–Cahn/Cahn–Hilliard system, that are discrete in space and continuous in time. Thus, we consider the following one-dimensional spatially discrete coupled system:

$$\dot{U}_i = \begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \begin{pmatrix} -\beta_1 \Delta u_i - f_1(u_i, v_i) \\ -\alpha\Delta[-\beta_2 \Delta v_i - f_2(u_i, v_i)] \end{pmatrix} \quad \forall i \in \mathbb{Z}, \tag{1.1}$$

where $\beta_1 \neq 0$, $\alpha \neq 0$ and Δ denotes the discrete Laplacian operator defined by

$$\Delta u_i = u_{i+1} - 2u_i + u_{i-1}.$$

We take coupling functions f_1 and f_2 given by

$$f_1(u, v) = \begin{cases} (-\infty, -\gamma_1 + s_1 v], & u = -1, \\ \gamma_1 u + s_1 v, & |u| < 1, \\ [\gamma_1 + s_1 v, \infty), & u = 1, \\ \phi, & |u| > 1, \end{cases} \tag{1.2}$$

$$f_2(u, v) = \begin{cases} (-\infty, s_2 u - \gamma_2], & v = -1, \\ s_2 u + \gamma_2 u, & |v| < 1, \\ [s_2 u + \gamma_2, \infty), & v = 1, \\ \phi, & |v| > 1. \end{cases} \tag{1.3}$$

These are generalizations of the double obstacle nonlinearity [2,5,10,14,17]. Note that by fixing v in f_1 and varying u, or by fixing u in f_2 and varying v, we obtain a function similar to the double obstacle nonlinearity previously applied to spatially discrete Allen–Cahn equations [10] and Cahn–Hilliard equations [2]. The arguments of f_1 and f_2 are restricted to the interval $[-1, 1]$ with the values ± 1 acting as barriers. The parameters γ_1, γ_2, s_1 and s_2 can be positive, negative or zero. We assume, however, that s_1 and s_2 are not both zero as this would mean that f_1 and f_2 no longer behave as coupling functions, and we simply have separate Allen–Cahn and Cahn–Hilliard equations. We note here that Cahn and Novick-Cohen [8] focus on one of the more physically relevant parameter regions in which (1.1) has a well posed continuum limit, $\beta_1 < 0$ and $\alpha\beta_2 < 0$.

The equation for $\dot{u}_i$ corresponds to a one-dimensional spatially discrete version of the Allen–Cahn equation [10], which arises naturally from a spatial discretization of the partial differential equation

$$u_t = u_{xx} - f(u). \tag{1.4}$$

This discretization requires β_1 to be negative and of large norm, however, we are not restricted to the PDE case and hence consider a full range of parameters.

In a similar fashion, the equation for $\dot{v}_i$ is the one-dimensional spatially discrete Cahn–Hilliard equation which was introduced in [2]. When $\alpha > 0$ and $\beta_2 < 0$, this corresponds to a finite difference spatial discretization of the PDE

$$u_t = -(\varepsilon u_{xx} - f(u))_{xx} \quad \forall x \in \mathbb{R}, \tag{1.5}$$

but in what follows no restriction is placed on the sign of α and β_2 and the resulting system need not necessarily be near a PDE continuum limit.

Following the example in [10], subsequently used in [2], both systems are written with negative coupling coefficients.

Our motivation in studying spatially discrete models is threefold:

1. In many cases spatially discrete models allow for microscopic effects that cannot easily be modeled with continuum models. For example, anisotropy arises naturally in discrete models [7], also phenomena with fixed interaction length are easily modeled.
2. Often there is interesting dynamical behavior in spatially discrete models that is not present in the analogous continuum models. For example propagation failure of traveling waves arises and can be studied in discrete models [7,16] (spatially continuous models fail to represent this phenomenon). Also, discrete models are often applicable in parameter regions which are physically reasonable, but for which the PDE arising from the corresponding spatially continuous model is ill-posed.
3. In cases in which the spatially discrete model corresponds to the spatial discretization of a PDE model, careful study may lead to a better understanding of the effects of discretization. Note that (1.1) could be viewed as a finite difference discretization of the continuous Laplacian (where the mesh size h is factored into the parameters α, β_1 and β_2).

With f_1 and f_2 given by (1.2) and (1.3), the differential equations are interpreted as differential inclusions and the values of the variables are restricted to the ranges $|u_i| \le 1$ and $|v_i| \le 1$. The phase space of the system is hence

$$[-1,1]^{\mathbb{Z}} \times [-1,1]^{\mathbb{Z}} = \left\{ U = (u,v) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \;\middle|\; \begin{matrix} u_i \in [-1,1] & \forall i \in \mathbb{Z} \\ v_i \in [-1,1] & \forall i \in \mathbb{Z} \end{matrix} \right\}. \tag{1.6}$$

We add and subtract set-valued functions as follows. If $f = [a, \infty)$ and $g = [b, \infty)$ with $a, b \in \mathbb{R}$ then $f + g = [a + b, \infty)$. In other words $f + g$ is taken to be the set of all sums $x + y$, where $x \in f$ and $y \in g$. In particular, this implies that if f and g are both unbounded with opposite sign, i.e. $f(u, v) = [a, \infty)$ and $g(u, v) = (-\infty, b]$ with $a, b \in \mathbb{R}$, then their sum is the whole real line, $f + g = (-\infty, \infty)$.

Eq. (1.1) can now be written in the form

$$\begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \begin{pmatrix} -\beta_1(u_{i+1} - 2u_i + u_{i-1}) - f_1(u_i, v_i) \\ -\alpha[-\beta_2(v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}) \\ \quad -f_2(u_{i+1}, v_{i+1}) + 2f_2(u_i, v_i) - f_2(u_{i-1}, v_{i-1})] \end{pmatrix} \quad \forall i \in \mathbb{Z}, \tag{1.7}$$

both when f_1 and f_2 are uniquely valued, so that (1.7) has a conventional meaning as a differential equation, and in the case where f_1 and f_2 are set-valued so that (1.7) must be interpreted as a differential inclusion with the addition of sets on the right-hand side of (1.7) taking the meaning given above.

The time evolution of our system is described by an infinite system of ordinary differential equations that we call a *lattice differential equation* (see [9]). With $i \in \mathbb{Z}$ denoting the space variable, the state of our dynamical system is an infinite vector $\{U_i\}_{i\in\mathbb{Z}} = \{(u_i, v_i)\}_{i\in\mathbb{Z}}$. We are interested in bounded solutions and take $U \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, where $l^\infty(\mathbb{Z})$ is the Banach space with norm $\|\cdot\|_{l^\infty(\mathbb{Z})}$, given by

$$l^\infty(\mathbb{Z}) = \{u : \mathbb{Z} \to \mathbb{R} \,|\, \|u\|_{l^\infty(\mathbb{Z})} < \infty\}, \qquad \|u\|_{l^\infty(\mathbb{Z})} = \sup_{i\in\mathbb{Z}} |u_i|.$$

We write the general autonomous system $\dot{U} = g(U)$, where $g : l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, in coordinate form as

$$\dot{U}_i = g_i(\{U_j\}_{j\in\mathbb{Z}}) \quad \text{for } i \in \mathbb{Z}.$$

We can solve the initial value problem with $U(0) = U^0$ for any given $U^0 = (u^0, v^0) \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, if g is a locally Lipschitz function on $l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$. The familiar existence and uniqueness proof for finite dimensional ODEs carries over to this infinite dimensional setting (see, for example [18]). In Section 2, we provide conditions for existence and uniqueness of solutions of (1.1)–(1.3). These conditions do not depend on the parameters β_1, γ_2 or s_1. However, for $\alpha > 0$ we require $\beta_2 > 0$, and for $\alpha < 0$ we are restricted to a range of the parameters β_2, γ_2, and s_2.

In Section 3, we study equilibrium solutions of (1.1)–(1.3). We define mosaic solutions to be equilibrium solutions that are restricted to take only the values $+1$, -1, and 0, as in [2,10], and in Theorem 3.1 we give explicit necessary and sufficient conditions for an equilibrium solution to be a mosaic solution. Theorem 3.1 is in the spirit of results for discrete Allen–Cahn equations [10] and Cahn–Hilliard equations [2]. However, with the result presented here we obtain an understanding of the effect of the coupling through the nonlinear term and we obtain the results from Abell et al. [2,10] as special cases.

In Section 4, we consider the stability of mosaic solutions. Using the notion of weak forward invariance we obtain criteria for weakly Lyapunov stable and weakly asymptotically stable mosaic solutions, and identify classes of such mosaic solutions.

In Section 5, we introduce the concepts of pattern formation and spatial chaos. Using transition matrices we extend the analysis in [2,10], and determine the spatial entropy over certain ranges of parameter values for (1.1). We first calculate the spatial entropy of an essentially decoupled system. This system is much simpler than the coupled case, and so we can calculate the entropy explicitly for all parameter regions. We then extend this to calculate the entropy of various examples of parameter regions in the fully coupled case. This involves transitions between two 4-tuples (which overlap to form three 5-tuples) which arises from the 2-tuples for the Allen–Cahn equations [10] and the 4-tuples in the Cahn–Hilliard case [2]. We then focus on the effect of the coupling through the nonlinear term so that parameter ranges may be determined in which the equations act in an uncoupled fashion, when one equation drives the other, and when there is strong coupling but neither equation drive the other.

Finally, numerical simulations are presented in which the solution to the initial value problem is approximated. These numerical results confirm the importance of the analytical results by showing how the asymptotic state of the system depends upon the given parameter values.

2. Existence and uniqueness

We will establish existence and uniqueness theorems for the initial value problem (1.1)–(1.3). Because the functions f_1 and f_2 defined by (1.2) and (1.3) are set-valued, the theorems that we present are non-standard. In addition, our results concern only forward time, as existence and uniqueness do not hold in general for such systems in backward time. Our treatment will closely follow that in [2,10], where similar theorems were established for spatially discrete Allen–Cahn and Cahn–Hilliard equations.

Before proceeding, we must first define what we mean by a solution of such a system.

Definition 2.1. By a *solution* of (1.1)–(1.3), we mean a continuous function

$$U : I \to [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}} \subseteq l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$$

on some interval I, such that the coordinate function $U_i(t) = (u_i(t), v_i(t))$ is absolutely continuous in I for each $i \in \mathbb{Z}$, and such that the inclusions

$$\dot{u}_i(t) \in -\beta_1 \Delta u_i(t) - f_1(u_i(t), v_i(t)) \;\; \forall i \in \mathbb{Z}, \qquad \dot{v}_i(t) \in -\alpha \Delta[-\beta_2 \Delta v_i(t) - f_2(u_i(t), v_i(t))] \quad \forall i \in \mathbb{Z}, \tag{2.1}$$

hold for almost every $t \in I$.

Alternatives to Definition 2.1 are possible. In particular, for the Cahn–Hilliard term some authors prefer

$$\dot{v}_i(t) \in -\alpha \Delta w_i(t) \quad \forall i \in \mathbb{Z}, \qquad w_i(t) \in -\beta_2 \Delta v_i(t) - f_2(u_i(t), v_i(t)) \quad \forall i \in \mathbb{Z}.$$

This reduction of the fourth-order equation to a second-order equation effectively forces consistent choices of the set-valued f when evaluating the $\dot{v}_i$'s, which ensures conservation of mass in the Cahn–Hilliard component (under suitable boundary conditions). However, we prefer (2.1), even though it is not mass conserving in the Cahn–Hilliard component, because it is amenable to a purely local analysis which is not applicable to the reduced order equations.

We have the following existence result in forward time.

Theorem 2.1. *Consider* (1.1)–(1.3) *with* $U^0 \in [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$ *given. If*:

1. $\alpha > 0$ *and* $\beta_2 \geq 0$ *or*
2. $\alpha < 0$ *and* $\gamma_2 + |s_2| \leq 4\beta_2 \leq 0$,

then there exists a solution $U = (u, v) : [0, \infty) \times [0, \infty) \to [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$ *in forward time, to the initial value problem* $U(0) = U^0$, *or in other words* $(u(0), v(0)) = (u^0, v^0)$.

Proof. We will prove the result by constructing solutions to a series of approximating problems, to which a standard existence result applies, and then taking limits of the approximating solutions, after having obtained the appropriate a priori estimates.

The approximating problem is given by replacing the set-valued nonlinearities (1.2) and (1.3) with

$$f_1^\varepsilon(u, v) = \begin{cases} \dfrac{1}{\varepsilon}(u+1) - \gamma_1 + s_1 v & \text{if } u \leq -1, \\ \gamma_1 u + s_1 v & \text{if } |u| < 1, \\ \dfrac{1}{\varepsilon}(u-1) + \gamma_1 + s_1 v & \text{if } u \geq 1, \end{cases} \tag{2.2}$$

$$f_2^\varepsilon(u, v) = \begin{cases} \dfrac{1}{\varepsilon}(v+1) - \gamma_2 + s_2 u & \text{if } v \leq -1, \\ \gamma_2 v + s_2 u & \text{if } |v| < 1, \\ \dfrac{1}{\varepsilon}(v-1) + \gamma_2 + s_2 u & \text{if } v \geq 1, \end{cases} \tag{2.3}$$

which for any $\varepsilon \neq 0$ are globally Lipschitz functions $f_1^\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f_2^\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. With $\varepsilon > 0$ fixed, consider (1.1) with f_1^ε and f_2^ε replacing f_1 and f_2. We shall restrict ε to be sufficiently small, specifically,

$$|\gamma_1|\varepsilon \leq 1, \qquad |\gamma_2|\varepsilon \leq 1, \tag{2.4}$$

$$(8|\beta_1| + 2|s_1|)\varepsilon \leq 1, \qquad (8|\beta_2| + 2|s_2|)\varepsilon \leq 1. \tag{2.5}$$

We can write this system, with the nonlinearities f_1^ε and f_2^ε, abstractly as an ordinary differential equation $\dot{U}^\varepsilon = F^\varepsilon(U^\varepsilon)$, with $U^\varepsilon = (u^\varepsilon, v^\varepsilon)$. This equation is in the Banach space $l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, and $F^\varepsilon : l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z}) \to$

$l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ is a globally Lipschitz function. In a standard fashion we obtain a unique solution $U^\varepsilon = (u^\varepsilon, v^\varepsilon) : \mathbb{R} \times \mathbb{R} \to l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ to the initial value problem $U^\varepsilon(0) = U^0$, with $\|U^0\|_{l^\infty(\mathbb{Z})} \le 1$ as in the statement of Theorem 2.1. In other words, $(u(0), v(0)) = (u^0, v^0)$ with $\|u^0\|_{l^\infty(\mathbb{Z})} \le 1$ and $\|v^0\|_{l^\infty(\mathbb{Z})} \le 1$.

Observe that this solution satisfies

$$\dot{u}_i^\varepsilon = -\beta_1 \Delta u_i^\varepsilon - \gamma_1 u_i^\varepsilon - s_1 v_i^\varepsilon - h_i^{1,\varepsilon}(t) \ \ \forall i \in \mathbb{Z}, \quad \dot{v}_i^\varepsilon = -\alpha \Delta[-\beta_2 \Delta v_i^\varepsilon - \gamma_2 v_i^\varepsilon - s_2 u_i^\varepsilon - h_i^{2,\varepsilon}(t)] \quad \forall i \in \mathbb{Z}, \tag{2.6}$$

where the continuous functions $h_i^{1,\varepsilon} : \mathbb{R} \to \mathbb{R}$ and $h_i^{2,\varepsilon} : \mathbb{R} \to \mathbb{R}$ are given by

$$h_i^{1,\varepsilon} = f_1^\varepsilon(u_i^\varepsilon(t), v_i^\varepsilon(t)) - (\gamma_1 u_i^\varepsilon(t) + s_1 v_i^\varepsilon(t)), \qquad h_i^{2,\varepsilon} = f_2^\varepsilon(u_i^\varepsilon(t), v_i^\varepsilon(t)) - (\gamma_2 v_i^\varepsilon(t) + s_2 u_i^\varepsilon(t)),$$

and satisfy

$$h_i^{1,\varepsilon}(t) \begin{cases} \le 0 & \text{if } u_i^\varepsilon(t) \le -1, \\ = 0 & \text{if } |u_i^\varepsilon(t)| < 1, \\ \ge 0 & \text{if } u_i^\varepsilon(t) \ge 1, \end{cases} \qquad h_i^{2,\varepsilon}(t) \begin{cases} \le 0 & \text{if } v_i^\varepsilon(t) \le -1, \\ = 0 & \text{if } |v_i^\varepsilon(t)| < 1, \\ \ge 0 & \text{if } v_i^\varepsilon(t) \ge 1, \end{cases} \tag{2.7}$$

where here we have used the bounds of (2.4) on ε.

We now establish the uniform bounds

$$\begin{aligned} &|u_i^\varepsilon(t)| \le 1 + K_1\varepsilon \quad \forall i \in \mathbb{Z} \ \forall t \ge 0, \qquad |v_i^\varepsilon(t)| \le 1 + K_1\varepsilon \quad \forall i \in \mathbb{Z} \ \forall t \ge 0, \\ &|h_i^{1,\varepsilon}(t)| \le +K_2 \quad \forall i \in \mathbb{Z} \ \forall t \ge 0, \qquad |h_i^{2,\varepsilon}(t)| \le K_2 \quad \forall i \in \mathbb{Z} \ \forall t \ge 0 \end{aligned} \tag{2.8}$$

with constants K_1 and K_2 independent of ε, as well as of i and t. (However, the constants K_1 and K_2 generally depend on $\beta_1, \beta_2, \gamma_2, \gamma_2, s_1$ and s_2.)

First observe that if $1 \le u_i^\varepsilon(t) \le 1 + K_1\varepsilon$, then from (2.6) and the formula (2.2) for f_1^ε, we have from the definition of $h_i^{1,\varepsilon}(t)$ that

$$|h_i^{1,\varepsilon}(t)| = \left| \left(\frac{1}{\varepsilon} - \gamma_1 \right) (u_i^\varepsilon(t) - 1) \right| \le K_1 |1 - \gamma_1\varepsilon| \le 2K_1$$

by (2.4). The same bound holds for $-1 - K_1\varepsilon \le u_i^\varepsilon(t) \le -1$, and of course $h_i^{1,\varepsilon}(t) = 0$ if $|u_i^\varepsilon(t)| < 1$. Similar bounds hold for $h_i^{2,\varepsilon}(t)$, and hence the bounds in (2.8) on $h_i^{1,\varepsilon}(t)$ and $h_i^{2,\varepsilon}(t)$ hold with $K_2 = 2K_1$.

To bound $u_i^\varepsilon(t)$ and $v_i^\varepsilon(t)$, we show that the ball $\{U^\varepsilon \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z}) : \|U^\varepsilon\|_{l^\infty(\mathbb{Z})} \le 1 + K_1\varepsilon\}$ is positively invariant. To do this it is sufficient to prove that

$$\dot{u}_i^\varepsilon(t) \le 0 \quad \text{whenever } u_i^\varepsilon(t) = 1 + K_1\varepsilon, \quad \text{and} \quad \|U^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon, \tag{2.9}$$

$$\dot{v}_i^\varepsilon(t) \le 0 \quad \text{whenever } v_i^\varepsilon(t) = 1 + K_1\varepsilon, \quad \text{and} \quad \|U^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon, \tag{2.10}$$

along with the corresponding inequalities $\dot{u}_i^\varepsilon(t) \ge 0$, whenever $u_i^\varepsilon(t) = -1 - K_1\varepsilon$ and $\|U^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon$ and also $\dot{v}_i^\varepsilon(t) \ge 0$, whenever $v_i^\varepsilon(t) = -1 - K_1\varepsilon$ and $\|U^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon$.

We prove (2.9) and (2.10), as the proofs for the corresponding inequalities at $u_i^\varepsilon(t) = -1 - K_1\varepsilon$ and $v_i^\varepsilon(t) = -1 - K_1\varepsilon$ are similar.

First, let $u_i^\varepsilon(t) = 1 + K_1\varepsilon$, and $\|U^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon$ and assume $K_1 \ge 2(4|\beta_1| - \gamma_1 + |s_1|)$. Then

$$\begin{aligned} \dot{u}_i^\varepsilon &= -\beta_1(u_{i+1}^\varepsilon - 2u_i^\varepsilon + u_{i-1}^\varepsilon) - \left[\frac{1}{\varepsilon}(u_i^\varepsilon - 1) + \gamma_1 + s_1 v_i^\varepsilon \right] \le 4|\beta_1|(1 + K_1\varepsilon) - K_1 - \gamma_1 + |s_1|(1 + K_1\varepsilon) \\ &= 4|\beta_1| + |s_1| + K_1\varepsilon(4|\beta_1| + |s_1|) - K_1 - \gamma_1 \le 4|\beta_1| + |s_1| - \frac{K_1}{2} - \gamma_1 \le 0, \end{aligned}$$

where here we have used $(8|\beta_1|+2|s_1|)\varepsilon \le 1$ from (2.5), as well as the assumption that $K_1 \ge 2(4|\beta_1|-\gamma_1+|s_1|)$. This establishes (2.9).

Second, assume $v_i^\varepsilon(t) = 1 + K_1\varepsilon$ and $\|U^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} = 1 + K_1\varepsilon$. Let

$$g^\varepsilon(u, v) = f_2^\varepsilon(u, v) - 4\beta_2 v. \tag{2.11}$$

Then from (1.1) and (2.3), we have

$$\dot{v}_i^\varepsilon = \alpha[\beta_2(6v_i^\varepsilon + v_{i+2}^\varepsilon + v_{i-2}^\varepsilon) + g^\varepsilon(u_{i+1}^\varepsilon, v_{i+1}^\varepsilon) - 2f_2^\varepsilon(u_i^\varepsilon, v_i^\varepsilon) + g^\varepsilon(u_{i-1}^\varepsilon, v_{i-1}^\varepsilon)]. \tag{2.12}$$

The following inequalities will be useful. If $\gamma_2 + |s_2| \ge 4\beta_2$ or $4\beta_2 \ge \gamma_2 + |s_2|$ and

$$K_1 \ge \frac{2(4\beta_2 - \gamma_2 - |s_2|)}{1 - 4\varepsilon\beta_2 + |s_2|\varepsilon}, \tag{2.13}$$

then if $\|u^\varepsilon(t)\|_{l^\infty(\mathbb{Z})} \le 1 + K_1\varepsilon$ it follows that

$$g^\varepsilon(u_j, v_j) \le \gamma_2 - 4\beta_2 + |s_2| + K_1(1 - 4\varepsilon\beta_2 + |s_2|\varepsilon) \tag{2.14}$$

with the right-hand side non-negative (using (2.5)). It also follows from (2.3) and (2.11) that if $4\beta_2 \ge \gamma_2$ and

$$K_1 \le \frac{2(4\beta_2 - \gamma_2 - |s_2|)}{1 - 4\varepsilon\beta_2 + |s_2|\varepsilon}, \tag{2.15}$$

then

$$\gamma_2 - 4\beta_2 + |s_2| \le g^\varepsilon(u_j, v_j) \le 4\beta_2 - \gamma_2 + |s_2|. \tag{2.16}$$

We use (2.11) to prove (2.10) in the two different cases.

1. $\alpha > 0$ and $\beta_2 \ge 0$.
   Let $K_1 > 0$ satisfy (2.13). Thus, if $u_i^\varepsilon = 1 + K_1\varepsilon$ and $a > 0$, it follows from (2.12) that
   $$\begin{aligned}\dot{v}_i^\varepsilon &\le 2\alpha[3\beta_2(1 + K_1\varepsilon) + |\beta_2|(1 + K_1\varepsilon) + \gamma_2 - 4\beta_2 + |s_2| + K_1(1 - 4\varepsilon\beta_2 + |s_2|\varepsilon)\\ &\quad -(K_1 + \gamma_2 + |s_2|(1 + K_1\varepsilon))] = 2\alpha[(|\beta_2| - \beta_2)(1 + K_1\varepsilon)] = 0,\end{aligned}$$
   provided $\beta_2 \ge 0$. This establishes (2.9).
2. $\alpha < 0$ and $\gamma_2 + |s_2| \le 4\beta_2 \le 0$.
   Let $K_1 = 0$. Thus (2.15) is satisfied and hence (2.16) holds. Thus, if $u_i^\varepsilon = 1$ and $\alpha < 0$, it follows from (2.12) that
   $$\dot{v}_i^\varepsilon \le 2\alpha[3\beta_2|\beta_2| + \gamma_2 - 4\beta_2 + |s_2| - \gamma_2 - |s_2|] = 2\alpha[-(\beta_2 + |\beta_2|)] = 0,$$
   provided $\beta_2 \le 0$. This establishes (2.10).

The proof of the corresponding inequalities at $-1 - K_1\varepsilon$ are similar in both cases. Thus the ball $\{U^\varepsilon \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z}) : \|U\|_{l^\infty(\mathbb{Z})} \le 1 + K_1\varepsilon\}$ is positively invariant, which establishes the remaining bounds on $|u_i^\varepsilon(t)|$ and $|v_i^\varepsilon(t)|$ in (2.8).

From (2.8), we obtain from the differential equation (2.6) the additional bounds on the derivative of $U_i^\varepsilon(t) = (u_i^\varepsilon(t), v_i^\varepsilon(t))$ that $|\dot{u}_i^\varepsilon(t)| \le K_3$ and $|\dot{v}_i^\varepsilon(t)| \le K_3$, for some K_3 valid for all i, non-negative t, and $\varepsilon \le 1$. Upon taking a sequence $\varepsilon_n \to 0$, and possibly passing to a subsequence, we have with a standard application of Ascoli's theorem, the limits

$$U_i^{\varepsilon_n}(t) \to U_i(t) \quad \text{uniformly for } 0 \le t \le T,$$

in other words that

$$u_i^{\varepsilon_n}(t) \to u_i(t) \quad \text{uniformly for } 0 \le t \le T, \qquad v_i^{\varepsilon_n}(t) \to v_i(t) \quad \text{uniformly for } 0 \le t \le T,$$

and also

$$h_i^{1,\varepsilon_n}(t) \to h_i^1(t) \text{ weak}^* \quad \text{in } L^\infty(0,T), \qquad h_i^{2,\varepsilon_n}(t) \to h_i^2(t) \text{ weak}^* \quad \text{in } L^\infty(0,T)$$

for each $i \in \mathbb{Z}$ and $T > 0$. The limiting functions $U_i(t) = (u_i(t), v_i(t))$ are absolutely continuous, enjoy the bounds $|u_i(t)| \le 1$ and $|v_i(t)| \le 1$ for all $t \ge 0$, and satisfy the initial condition $U(0) = U^0$. They also satisfy

$$\dot{u}_i = -\beta_1 \Delta u_i - \gamma_1 u_i - s_1 v_i - h_i(t) \quad \forall i \in \mathbb{Z}, \qquad \dot{v}_i = -\alpha\Delta[-\beta_2 \Delta v_i - \gamma_2 v_i - s_2 u_i - h_i(t)] \quad \forall i \in \mathbb{Z} \tag{2.17}$$

for almost every $t \ge 0$, as one sees by integrating Eq. (2.6) from 0 to any $t > 0$, and taking the limit $\varepsilon_n \to 0$. Finally, one sees from (2.7) that the functions $h_i^1(t)$ and $h_i^2(t)$ satisfy

$$h_i^1(t) \begin{cases} \le 0 & \text{if } u_i(t) = -1, \\ = 0 & \text{if } |u_i(t)| < 1, \\ \ge 0 & \text{if } u_i(t) = 1, \end{cases} \qquad h_i^2(t) \begin{cases} \le 0 & \text{if } v_i(t) = -1, \\ = 0 & \text{if } |v_i(t)| < 1, \\ \ge 0 & \text{if } v_i(t) = 1 \end{cases} \tag{2.18}$$

for almost every $t \ge 0$. With this, it is now clear that $U = (u, v) : [0,\infty) \times [0,\infty) \to [-1,1]^{\mathbb{Z}} \times [-1,1]^{\mathbb{Z}}$ is a solution to (1.1)–(1.3) with the initial condition $U(0) = U^0$, as desired. We note in particular, that the uniform bounds $|\dot{u}_i(t)| \le K_3$ and $|\dot{v}_i(t)| \le K_3$ ensure the continuity of $u(t)$ and $v(t)$ in t, as elements of $l^\infty(\mathbb{Z})$. □

We will consider existence and stability of mosaic solutions for all parameter values, not just those that satisfy Theorem 2.1. In the case of parameter values which do not satisfy Theorem 2.1, we may not have infinite time existence of solutions for arbitrary initial conditions in $[-1,1]^{\mathbb{Z}} \times [-1,1]^{\mathbb{Z}}$. However, since a mosaic solution is an equilibrium solution, we will always have infinite time existence for these solutions. Our first step to establishing stability of mosaic solutions will be to show that a neighborhood of the mosaic solution is forward invariant. In this case, existence of solutions for all initial conditions within the neighborhood will follow using the techniques above, and indeed for any initial conditions within the basin of attraction of the forward invariant neighborhood.

We now prove that the solution constructed above is unique provided $\|U(t)\|_{l^\infty(\mathbb{Z})} < 1$. The Laplacian operator acting on the set-valued function f_2 for the v component precludes uniqueness in general, when $|v_i(t)| = 1$ for some i.

Theorem 2.2. *Let $U^1, U^2 : [0,\infty) \times [0,\infty) \to [-1,1]^{\mathbb{Z}} \times [-1,1]^{\mathbb{Z}}$ be two solutions of* (1.1)–(1.3) *with the same initial condition $U^1(0) = U^2(0) = U^0$, and with $U^1(t), U^2(t) \in (-1,1)^{\mathbb{Z}} \times (-1,1)^{\mathbb{Z}}$ for all $t \in [0,\tau]$ for some $\tau > 0$. Then $U^1(t) = U^2(t)$ for all $t \in [0,\tau]$.*

Proof. Consider (1.1) with

$$f_1^\gamma(u,v) = \gamma_1 u + s_1 v \quad \forall (u,v) \in \mathbb{R} \times \mathbb{R}, \qquad f_2^\gamma(u,v) = \gamma_2 v + s_2 u \quad \forall (u,v) \in \mathbb{R} \times \mathbb{R}, \tag{2.19}$$

replacing f_1 and f_2. Writing this system abstractly as an ordinary differential equation $\dot{U}^\gamma = F^\gamma(U^\gamma)$ in the Banach space $l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, where $F^\gamma : l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ is a globally Lipschitz function, as noted in Section 1, in a standard fashion we obtain a unique solution $U^\gamma : \mathbb{R} \times \mathbb{R} \to l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ to the initial value problem $U^\gamma(0) = U^0$, and the result follows. □

It is straightforward to show that $|v_i| = 1$ can cause non-uniqueness in the solution for the neighboring points on the lattice, as we now illustrate. Consider (1.1)–(1.3) with $\alpha > 0$ and $(3\beta_2 - \gamma_2) > 0$ and initial condition

$U_0(0) = (0, \hat{v})$ and $U_j(0) = (0, 0)$ for all $j \neq 0$. Then if $\hat{v} \in (1-\delta, 1)$, we have $\dot{v}_0(0) = 2\alpha(3\beta_2 - \gamma_2)\hat{v} > 0$. But now since $\dot{v}_i$ satisfies (2.17) there exists some interval $[0, \varepsilon)$ such that $\dot{v}_0(t) > 0$ for all $t \in [0, \varepsilon)$ provided $\dot{v}_0(t) < 1$. It follows that if we set $\hat{v} = 1$, then any solution of (1.1)–(1.3) must satisfy $v_0(t) = 1$ for all $t \in [0, \varepsilon)$, and moreover, Theorem 2.1 guarantees that at least one such solution exists. Now consider $v_1(t)$. We have $v_1(0) = 0$ and $\dot{v}_1(0) = \alpha(-4\beta_2 + [\gamma_2, \infty))$, and hence is interval valued. Moreover, since $v_0(t) = 1$ for all $t \in [0, \varepsilon)$, we have $\dot{v}_i(t)$ set-valued for all $t \in [0, \varepsilon)$ and non-uniqueness of the v component of the solution follows.

3. Equilibrium solutions

Definition 3.1. A function (u, v) with $u \in [-1, 1]^{\mathbb{Z}}$ and $v \in [-1, 1]^{\mathbb{Z}}$, is said to be an *equilibrium solution* of (1.1)–(1.3) if

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} -\beta_1 \Delta u_i - f_1(u_i, v_i) \\ -\alpha\Delta[-\beta_2 \Delta v_i - f_2(u_i, v_i)] \end{pmatrix} \quad \forall i \in \mathbb{Z}. \tag{3.1}$$

Note that this varies slightly from the standard definition of an equilibrium solution, which requires $(\dot{u}_i, \dot{v}_i) = (0, 0)$ for all $i \in \mathbb{Z}$ [20]. This difference is due to the set-valued nature of the functions f_1 and f_2 which mean that the equations in (1.1) are interpreted as differential inclusions. Clearly, if both $|u| < 1$ and $|v| < 1$ for all $i \in \mathbb{Z}$ then f_1 and f_2 are no longer set-valued and Definition 3.1 will agree with the standard definition in this case.

3.1. Constant solutions

Let $(u_i, v_i) = (\mu, \nu)$ for all $i \in \mathbb{Z}$, where $\mu, \nu \in (-1, 1)$. This implies that $|u_i| < 1$ and $|v_i| < 1$ for all $i \in \mathbb{Z}$ and so (1.1) can be written as

$$\begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \begin{pmatrix} -\beta_1(\mu - 2\mu + \mu) - \gamma_1\mu - s_1\nu \\ -\alpha[-\beta_2(2\nu - 8\nu + 6\nu) - \gamma_2(\nu - 2\nu + \nu) - s_2(\mu - 2\mu + \mu)] \end{pmatrix} = \begin{pmatrix} -\gamma_1\mu - s_1\nu \\ 0 \end{pmatrix},$$

and hence the constant solution $(u_i, v_i) = (\mu, -\gamma_1\mu/s_1)$ for all $i \in \mathbb{Z}$ is an equilibrium solution of (1.1)–(1.3) for $|\mu| < \min(1, |s_1|/|\gamma_1|)$. Note that in particular this implies that the zero solution $(u_i, v_i) = (0, 0)$ for all $i \in \mathbb{Z}$ is an equilibrium solution of (1.1)–(1.3) for any set of parameter values.

3.2. Mosaic solutions

Definition 3.2. An equilibrium solution of (1.1)–(1.3) is called a *mosaic solution* if $u_i \in \{-1, 0, 1\}$ and $v_i \in \{-1, 0, 1\}$ for all $i \in \mathbb{Z}$. Any function $u : \mathbb{Z} \to \{-1, 0, 1\}^{\mathbb{Z}}$ is a *one-dimensional mosaic*, and the set of all such mosaics is denoted $\mathcal{M}_1 = \{-1, 0, 1\}^{\mathbb{Z}}$. We will denote the set of all pairs of mosaics (u, v) by $\mathcal{M}_1^2 = \mathcal{M}_1 \times \mathcal{M}_1$.

For any $(u, v) \in \mathcal{M}_1^2$, define

$$\sigma_i^{j,u} := u_{i+j} + u_{i-j} \quad \forall i \in \mathbb{Z}, \quad j \in \mathbb{N}, \qquad \sigma_i^{j,v} := v_{i+j} + v_{i-j} \quad \forall i \in \mathbb{Z}, \quad j \in \mathbb{N}. \tag{3.2}$$

Using this notation, we now prove the following theorem that establishes necessary and sufficient conditions for the existence of mosaic equilibrium solutions of (1.1)–(1.3).

Theorem 3.1. *A mosaic* $(u, v) \in \mathcal{M}_1^2$ *is an equilibrium solution of* (1.1)–(1.3), *that is* (u, v) *is a mosaic solution, if and only if for the Allen–Cahn component either*

$$u_i = 0, \qquad \beta_1\sigma_i^{1,u} + s_1 v_i = 0 \tag{3.3}$$

or

$$u_i \neq 0, \qquad \beta_1(2 - \sigma_i^{1,u} u_i) - s_1 u_i v_i \geq \gamma_1, \tag{3.4}$$

and for the Cahn–Hilliard component one of the following holds:

1. $v_i = v_{i-1} = v_{i+1} = 0$ *and*
   $$\beta_2 \sigma_i^{2,v} + s_2(\sigma_i^{1,u} - 2u_i) = 0, \tag{3.5}$$
2. $v_i = 0$ *and* $v_{i+1} v_{i-1} = -1$,
3. $v_i v_{i\pm1} = 1$,
4. $2v_i - \sigma_i^{1,v} \neq 0$ *and*
   $$\beta_2 \left(4 - \frac{2v_i - \sigma_i^{2,v}}{2v_i - \sigma_i^{1,v}}\right) - s_2 \left(\frac{2u_i - \sigma_i^{1,u}}{2v_i - \sigma_i^{1,v}}\right) \geq \gamma_2 \tag{3.6}$$
   for each $i \in \mathbb{Z}$.

Proof. By expanding (3.1), it is clear that (u, v) is a mosaic solution if and only if

$$-\beta_1(\sigma_i^{1,u} - 2u_i) \in f_1(u_i, v_i) \quad \forall i \in \mathbb{Z}, \tag{3.7}$$

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) \in f_2(u_{i+1}, v_{i+1}) - 2f_2(u_i, v_i) + f_2(u_{i-1}, v_{i-1}) \quad \forall i \in \mathbb{Z}. \tag{3.8}$$

1. First consider Eq. (3.7). Note that since varying v_i does not change the nature of the function f_1, we need only consider explicitly the possible values of u_i. There will be two separate cases, $u_i = 0$ and $u_i \neq 0$.
   1.1. Suppose $u_i = 0$. Then $f_1(u_i, v_i) = s_1 v_i$ and hence $\dot{u}_i = \beta_1 \sigma_i^{1,u} + s_1 v_i$. Therefore $\dot{u}_i = 0$ if and only if $\beta_1 \sigma_i^{1,u} + s_1 v_i = 0$, which is Eq. (3.3).
   1.2. Now consider $u_i \neq 0$. If $u_i = 1$ then $f_1(u_i, v_i) = [\gamma_1 + s_1 v_i, \infty)$, and so $0 \in \dot{u}_i$ provided
   $$-\beta_1(\sigma_i^{1,u} - 2u_i) \geq \gamma_1 + s_1 v_i,$$
   or, equivalently,
   $$\beta_1(2u_i - \sigma_i^{1,u}) - s_1 v_i \geq \gamma_1. \tag{3.9}$$
   On the other hand, if $u_i = -1$ then $f_1(u_i, v_i) = (-\infty, -\gamma_1 + s_1 v_i]$ and we require
   $$\beta_1(2u_i - \sigma_i^{1,u}) - s_1 v_i \leq -\gamma_1. \tag{3.10}$$
   Hence, multiplying inequalities (3.9) and (3.10) by u_i, we obtain the single condition that for $u_i \neq 0$, we have $0 \in \dot{u}_i$ if and only if
   $$\beta_1(2 - \sigma_i^{1,u} u_i) - s_1 u_i v_i \geq \gamma_1,$$
   which is (3.4) in the statement of the theorem.
2. Now consider Eq. (3.8). In this case, varying u_i and $u_{i\pm1}$ does not affect the nature of the function $f_2(u, v)$, and so we only have to examine the various values of v_i and $v_{i\pm1}$ separately.
   2.1. First assume $v_i = 0$, which implies $f_2(u_i, v_i) = s_2 u_i$, and hence $0 \in \dot{v}_i$ provided
   $$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v}) + 2s_2 u_i \in f_2(u_{i+1}, v_{i+1}) + f_2(u_{i-1}, v_{i-1}). \tag{3.11}$$
   We treat the cases $\sigma_i^{1,v} = 0$ and $\sigma_i^{1,v} \neq 0$ separately.
   2.1.1. First suppose $\sigma_i^{1,v} = 0$.

If $v_{i+1} = v_{i-1} = 0$, then

$$f_2(u_{i+1}, v_{i+1}) + f_2(u_{i-1}, v_{i-1}) = s_2 u_{i+1} + s_2 u_{i-1} = s_2 \sigma_i^{1,u},$$

and so $0 \in \dot{u}_i$ if and only if $-\beta_2 \sigma_i^{2,v} + 2 s_2 u_i = s_2 \sigma_i^{1,u}$ which is equivalent to

$$\beta_2 \sigma_i^{2,v} + s_2(\sigma_i^{1,u} - 2u_i) = 0,$$

which is case (1) in the statement of the theorem.

If $v_{i+1} = -v_{i-1} = \pm 1$, then

$$f_2(u_{i+1}, v_{i+1}) + f_2(u_{i-1}, v_{i-1}) = (-\infty, s_2 u - \gamma_2] + [s_2 u + \gamma_2, \infty) = (-\infty, \infty),$$

and so $0 \in \dot{v}_i$ is trivially satisfied. This corresponds to case (2) in the statement of the theorem.

2.1.2. Now suppose $\sigma_i^{1,v} \neq 0$. First consider $\sigma_i^{1,v} > 0$. Then

$$\begin{aligned} f_2(u_{i+1}, v_{i+1}) + f_2(u_{i-1}, v_{i-1}) &= [s_2 u_{i+1} + \gamma_2 v_{i+1}, \infty) + [s_2 u_{i-1} + \gamma_2 v_{i-1}, \infty) \\ &= [s_2 u_{i+1} + \gamma_2 v_{i+1} + s_2 u_{i-1} + \gamma_2 v_{i-1}, \infty) \\ &= [s_2 \sigma_i^{1,u} + \gamma_2 \sigma_i^{1,v}, \infty). \end{aligned}$$

Therefore, $0 \in \dot{v}_i$ if and only if

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v}) + 2 s_2 u_i \in [s_2 \sigma_i^{1,u} + \gamma_2 \sigma_i^{1,v}, \infty),$$

which is equivalent to

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v}) - s_2(\sigma_i^{1,u} - 2u_i) \geq \gamma_2 \sigma_i^{1,v}.$$

On the other hand, if we suppose $\sigma_i^{1,v} < 0$ we find that

$$f_2(u_{i+1}, v_{i+1}) + f_2(u_{i-1}, v_{i-1}) = (-\infty, -\gamma_2 \sigma_i^{1,v} + s_2 \sigma_i^{1,u}],$$

and so $0 \in \dot{v}_i$ if and only if

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v}) - s_2(\sigma_i^{1,u} - 2u_i) \leq \gamma_2 \sigma_i^{1,v}.$$

Hence, since we have supposed $\sigma_i^{1,v} \neq 0$, we have that $0 \in \dot{v}_i$ provided

$$\frac{\beta_2(4\sigma_i^{1,v} - \sigma_i^{2,v}) + s_2(2u_i - \sigma_i^{1,u})}{\sigma_i^{1,v}} \geq \gamma_2.$$

But for $v_i = 0$ (and recalling that $\sigma_i^{1,v} \neq 0$, so in particular $(2v_i - \sigma_i^{1,v}) \neq 0$), we have

$$\frac{\beta_2(4\sigma_i^{1,v} - \sigma_i^{2,v}) + s_2(2u_i - \sigma_i^{1,u})}{\sigma_i^{1,v}} = \beta_2 \left(4 - \frac{2v_i - \sigma_i^{2,v}}{2v_i - \sigma_i^{1,v}}\right) - s_2 \left(\frac{2u_i - \sigma_i^{1,u}}{2v_i - \sigma_i^{1,v}}\right),$$

and so this corresponds to case (4) in the statement of the theorem.

2.2. Now suppose that $v_i = 1$, which implies that $f_2(u_i, v_i) = [s_2 u_i + \gamma_2, \infty)$. Hence (3.8) becomes

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) \in f_2(u_{i+1}, v_{i+1}) - 2[s_2 u_i + \gamma_2 v_i, \infty) + f_2(u_{i-1}, v_{i-1}). \tag{3.12}$$

There are two separate cases to consider.

2.2.1. If $\sigma_i^{1,v} < 0$ or $\sigma_i^{1,v} = v_{i+1} = v_{i-1} = 0$ then $f_2(u_{i+1}, v_{i+1})$ and $f_2(u_{i-1}, v_{i-1})$ are each either equal to $s_2 u_{i\pm 1}$ or $(-\infty, -\gamma_2 + s_2 u_{i\pm 1}]$, and so

$$f_2(u_{i+1}, v_{i+1}) - 2f_2(u_i, v_i) + f_2(u_{i-1}, v_{i-1}) = (-\infty, \gamma_2(\sigma_i^{1,v} - 2) + s_2(\sigma_i^{1,u} - 2u_i)].$$

Hence by (3.12), for $0 \in \dot{v}_i$, we require

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) \in (-\infty, \gamma_2(\sigma_i^{1,v} - 2) + s_2(\sigma_i^{1,u} - 2u_i)]$$

or, equivalently,

$$-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - s_2(\sigma_i^{1,u} - 2u_i) \leq \gamma_2(\sigma_i^{1,v} - 2).$$

Since we are supposing $\sigma_i^{1,v} \leq 0$, we have $(\sigma_i^{1,v} - 2) < 0$ and hence

$$\frac{-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - s_2(\sigma_i^{1,u} - 2u_i)}{\sigma_i^{1,v} - 2} \geq \gamma_2.$$

But for $v_i = 1$, we have

$$\frac{-\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - s_2(\sigma_i^{1,u} - 2u_i)}{\sigma_i^{1,v} - 2} = \beta_2\left(4 - \frac{2v_i - \sigma_i^{2,v}}{2v_i - \sigma_i^{1,v}}\right) - s_2\left(\frac{2u_i - \sigma_i^{1,u}}{2v_i - \sigma_i^{1,v}}\right),$$

which again corresponds to case (4) in the statement of the theorem.

2.2.2. If $\sigma_i^{1,v} > 0$ or $\sigma_i^{1,v} = 0$ with $v_{i+1} = -v_{i-1} = \pm 1$, then at least one of the v_{i+1} or v_{i-1} is equal to 1, and so at least one of the $f_2(u_{i+1}, v_{i+1})$, $f_2(u_{i-1}, v_{i-1})$ is equal to $[s_2 u_{i\pm 1} + \gamma_2, \infty)$. It follows that

$$f_2(u_{i+1}, v_{i+1}) - 2f_2(u_i, v_i) + f_2(u_{i-1}, v_{i-1}) = (-\infty, \infty),$$

and so (3.12) is trivially satisfied. This corresponds to case (3) in the statement of the theorem.

2.3. Finally, the case $v_i = -1$ is similar to the case $v_i = 1$. This completes the proof. □

4. Stability of coupled equilibrium solutions

To study the stability of mosaic equilibrium solutions $(u, v) \in \mathcal{M}_1^2$, we will need to consider the behavior of solutions $(w, \bar{w}) \in [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$, which are perturbations of mosaic equilibrium solutions. For any $(w, \bar{w}) \in [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$, we define

$$\tau_i^{j,w} := w_{i+j} + w_{i-j} \quad \forall i \in \mathbb{Z},\ j \in \mathbb{N}, \qquad \tau_i^{j,\bar{w}} := \bar{w}_{i+j} + \bar{w}_{i-j} \quad \forall i \in \mathbb{Z},\ j \in \mathbb{N} \tag{4.1}$$

Now, for $(u, v) \in \mathcal{M}_1^2$ and $\delta, \theta > 0$, define the set

$$\mathcal{N}(u, v, \theta, \delta) = \left\{ (w, \bar{w}) : \mathbb{Z} \times \mathbb{Z} \to [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}} : \begin{array}{l} |w_i - u_i| \leq \theta \ \text{ if } \ u_i = 0 \\ |w_i - u_i| \leq \delta \ \text{ if } \ u_i = \pm 1 \\ |\bar{w}_i - v_i| \leq \theta \ \text{ if } \ v_i = 0 \\ |\bar{w}_i - v_i| \leq \delta \ \text{ if } \ v_i = \pm 1 \end{array} \right\}. \tag{4.2}$$

Thus $\mathcal{N}(u, v, \theta, \delta)$ defines a neighborhood of (u, v) in the phase space $[-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$. Note that for all $(w, \bar{w}) \in \mathcal{N}(u, v, \theta, \delta)$

$$\begin{aligned} &u_i = 1 \Rightarrow 1-\delta \le w_i \le 1, \qquad u_i = 0 \Rightarrow -\theta \le w_i \le \theta, \qquad u_i = -1 \Rightarrow -1 \le w_i \le -1+\delta, \\ &v_i = 1 \Rightarrow 1-\delta \le \bar{w}_i \le 1, \qquad v_i = 0 \Rightarrow -\theta \le \bar{w}_i \le 0, \qquad v_i = -1 \Rightarrow -1 \le \bar{w}_i \le -1+\delta, \end{aligned} \tag{4.3}$$

and hence defining

$$M := \max\{\theta, \delta\}, \tag{4.4}$$

we have

$$|w_i - u_i| \le M, \qquad |\bar{w}_i - v_i| \le M, \tag{4.5}$$

$$|\tau_i^{j,w} - \sigma_i^{j,u}| \le 2M, \qquad |\tau_i^{j,\bar{w}} - \sigma_i^{j,v}| \le 2M. \tag{4.6}$$

4.1. Weak-stability for differential inclusions

The set-valued nonlinearities (1.2) and (1.3) mean that we do not have uniqueness of solutions for (1.1)–(1.3), and hence we cannot use the standard definitions of asymptotic and Lyapunov stability for dynamical systems. Following [2] we make the following definition.

Definition 4.1. Let $(u, v) : [0, \infty) \times [0, \infty) \to [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$ be a solution of (1.1)–(1.3), with $(u(0), v(0)) = (u^0, v^0)$. Then

$$\Gamma(u^0, v^0) = \{(u(t), v(t)) : t \ge 0\} \tag{4.7}$$

is said to be a *forward orbit* of (u^0, v^0).

This forward orbit need not be unique, since we have a differential inclusion, rather than a dynamical system. This leads to new definitions of weak and strong forward invariance, asymptotic and Lyapunov stability. Here, the "strong" definition follows by requiring that every forward orbit of each point have the relevant property for forward invariance, asymptotic or Lyapunov stability, whereas the "weak" definition follows by requiring only that each point have a forward orbit with this property. As an illustration we reproduce the definitions of weak and strong Lyapunov stability here. For all the definitions, see [2] or [1].

Definition 4.2. Let $(u, v) \in [-1, 1]^{\mathbb{Z}} \times [-1, 1]^{\mathbb{Z}}$ be an equilibrium solution of (1.1)–(1.3) in the sense of Definition 3.1. Then (u, v) is *weakly Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $(w, \bar{w}) \in \mathcal{N}(u, v, \theta, \delta)$ has a forward orbit $\Gamma(w, \bar{w}) \subset \mathcal{N}(u, v, \theta, \delta)$. Moreover, (u, v) is *strongly Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every forward orbit $\Gamma(w, \bar{w})$ of every point $(w, \bar{w}) \in \mathcal{N}(u, v, \delta, \delta)$ satisfies $\Gamma(w, \bar{w}) \subset \mathcal{N}(u, v, \theta, \delta)$.

Note that the weak and strong concepts would be equivalent to each other and to the standard dynamical systems definitions if we had uniqueness of solutions.

We will often use the properties that if $\mathcal{N}(u, v, \theta, \delta)$ is weakly forward invariant for (1.1)–(1.3) with $\theta, \delta > 0$ arbitrarily small then (u, v) is weakly Lyapunov stable. If in addition every $(w^0, \bar{w}^0) \in \mathcal{N}(u, v, \theta, \delta)$ has a forward orbit which satisfies $(w(t), \bar{w}(t)) \to (u, v)$ as $t \to \infty$ then (u, v) is weakly asymptotically stable.

4.2. Stability of coupled mosaic solutions

The following inequalities and equalities:

$$\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i = 0, \tag{4.8}$$

$$u_i[\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i] > 0, \tag{4.9}$$

$$\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - \gamma_2(2v_i - \sigma_i^{1,v}) - s_2(2u_i - \sigma_i^{1,u}) = 0, \tag{4.10}$$

$$v_i\alpha[\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - \gamma_2(2v_i - \sigma_i^{1,v}) - s_2(2u_i - \sigma_i^{1,u})] > 0, \tag{4.11}$$

will be useful below and are analogous to the conditions (3.3)–(3.6), respectively, in Theorem 3.1 on the existence of mosaic equilibrium solutions. We now prove a lemma which will be used to establish weak-stability.

Lemma 4.3. *Let* $(u, v) \in \mathcal{M}_1^2$ *be a mosaic solution of* (1.1)–(1.3) *which satisfies* $u_i = \pm 1$, $v_i = \pm 1$ *for all* $i \in \mathbb{N}$. *If one of*:

1. (4.9),
2. (4.8), $2\beta_1 - \gamma_1 < 0$, $\beta_1 < 0$ *and* $4\beta_1 - \gamma_1 \le -|s_1|$,
3. (4.8), $2\beta_1 - \gamma_1 < 0$, $\beta_1 > 0$ *and* $\gamma_1 \ge |s_1|$,
   holds for each $i \in \mathbb{N}$, *and also either*
4. (4.11), *or*
5. (4.10), $0 \le 4\alpha\beta_2 \le \alpha\gamma_2$, $\alpha s_2 u_i v_i > 0$ *and* $u_i\sigma_i^{1,u} = -2$
   holds for each $i \in \mathbb{N}$, *then* $\mathcal{N}(u, v, \theta, \delta)$ *is weakly forward invariant for all sufficiently small* $\theta, \delta > 0$.

Proof. Assume throughout that $u_i = \pm 1$ and $v_i = \pm 1$ for all $i \in \mathbb{N}$. We first show that the result follows when conditions (1) and (4) of the theorem hold and subsequently show that the result follows when conditions (2) or (3) hold along with condition (5).

1. First, suppose that (4.9) and (4.11) hold. Let (u, v) be a mosaic solution of (1.1)–(1.3) satisfying Theorem 3.1. Let $(w, \bar{w})$ be a solution of (1.1)–(1.3), with $(w(0), \bar{w}(0)) \in \mathcal{N}(u, v, \theta, \delta)$.

   For $|w_i| \ne 1$, we can write the equation for $\dot{w}_i$ as

$$\dot{w}_i = \beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i + \beta_1[2(w_i - u_i) - (\tau_i^{1,w} - \sigma_i^{1,u})] - \gamma_1(w_i - u_i) - s_1(\bar{w}_i - v_i). \tag{4.12}$$

Hence,

$$\begin{aligned}|\dot{w}_i - [\beta_1(2u_i - \sigma_i^{j,u}) - \gamma_1 u_i - s_1 v_i]| &= |\beta_1[2(w_i - u_i) - (\tau_i^{1,w} - \sigma_i^{1,u})] - \gamma_1(w_i - u_i) - s_1(\bar{w}_i - v_i)| \\ &\le |w_i - u_i||2\beta_1 - \gamma_1| + |\tau_i^{1,w} - \sigma_i^{1,u}||\beta_1| + |\bar{w}_i - v_i||s_1| \\ &\le M|2\beta_1 - \gamma_1| + 2M|\beta_1| + M|s_1| \\ &= M[|2\beta_1 - \gamma_1| + 2|\beta_1| + |s_1|].\end{aligned}$$

Now by (4.9), we have $\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i \ne 0$ and hence, for M sufficiently small

$$M[|2\beta_1 - \gamma_1| + 2|\beta_1| + |s_1|] < |\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i|, \tag{4.13}$$

which implies

$$|\dot{w}_1 - [\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i]| \le |\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i|.$$

Hence, if $(w,\bar{w})\in\mathcal{N}(u,v,\theta,\delta)$, then $\dot{w}_i$ has the same sign as $\beta_1(2u_i-\sigma_i^{1,u})-\gamma_1u_i-s_1v_i$. Thus, from (4.9), we have $\dot{w}_i\geq 0$ when $u_i=1$ and $(w,\bar{w})\in\mathcal{N}(u,v,\theta,\delta)$, and $\dot{w}_i\leq 0$ when $u_i=-1$ and $(w,\bar{w})\in\mathcal{N}(u,v,\theta,\delta)$.

Now consider the equation for $\dot{\bar{w}}_i$:

$$\begin{aligned}\dot{\bar{w}}_i &=\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{1,u})]+\alpha[\beta_2((\tau_i^{2,\bar{w}}-\sigma_i^{2,v})\\&\quad-4(\tau_i^{1,\bar{w}}-\sigma_i^{1,v})+6(\bar{w}_i-v_i))-\gamma_2(2(\bar{w}_i-v_i)-(\tau_i^{1,\bar{w}}-\sigma_i^{1,v}))\\&\quad-s_2(2(w_i-u_i)-(\tau_i^{1,w}-\sigma_i^{1,u}))],\end{aligned}\tag{4.14}$$

which can be written in the form

$$\begin{aligned}\dot{\bar{w}}_i &=\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{j,u})]\\&\quad+\alpha[(\bar{w}_i-v_i)(6\beta_2-2\gamma_2)-(\tau_i^{1,\bar{w}}-\sigma_i^{1,v})(4\beta_2-\gamma_2)+\beta_2(\tau_i^{2,\bar{w}}-\sigma_i^{2,v})\\&\quad-s_2(2(w_i-u_i)-(\tau_i^{1,w}-\sigma_i^{1,u}))].\end{aligned}\tag{4.15}$$

Hence,

$$\begin{aligned}&|\dot{w}_i-\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{j,u})]|\\&\quad=\alpha[(\bar{w}_i-v_i)(6\beta_2-2\gamma_2)-(\tau_i^{1,\bar{w}}-\sigma_i^{1,v})(4\beta_2-\gamma_2)+\beta_2(\tau_i^{2,\bar{w}}-\sigma_i^{2,v})\\&\qquad-s_2(2(w_i-u_i)-(\tau_i^{1,w}-\sigma_i^{1,u}))]\leq M|\alpha|[|6\beta_2-2\gamma_2|+2|4\beta_2-\gamma_2|+2|\beta_2|+4|s_2|].\end{aligned}$$

By (4.11), we have $\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{1,u})]\neq 0$, which implies that for M sufficiently small

$$\begin{aligned}&M|\alpha|[|6\beta_2-2\gamma_2|+2|4\beta_2-\gamma_2|+2|\beta_2|+4|s_2|]\\&\quad<|\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{1,u})]|.\end{aligned}$$

Therefore, if $(w,\bar{w})\in\mathcal{N}(u,v,\theta,\delta)$, then $\dot{\bar{w}}_i$ has the same sign as

$$\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{1,u})],$$

and so from (4.11), we have $\dot{\bar{w}}_i\geq 0$ when $v_i=1$ and $(w,\bar{w})\in\mathcal{N}(u,v,\theta,\delta)$, and $\dot{\bar{w}}_i\leq 0$ when $v_i=-1$ and $(w,\bar{w})\in\mathcal{N}(u,v,\theta,\delta)$.

We can repeat the above arguments for all $i\in\mathcal{Z}$, since for $\beta_1,\gamma_1,s_1\in\mathbb{R}$ there are only finitely many non-zero values of $\beta_1(2u_i-\sigma_i^{1,u})-\gamma_1u_i-s_1v_i$ and similarly, for $\alpha,\beta_2,\gamma_2,s_2\in\mathbb{R}$ there are only finitely many non-zero values of

$$\alpha[\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{1,u})],$$

so that we can choose $M>0$. Hence, if parts (1) and (4) of Lemma 4.3 hold then $\mathcal{N}(u,v,\theta,\delta)$ is weakly forward invariant.

2. Now, suppose (4.8) and (4.10) hold, so that

$$\beta_1(2u_i-\sigma_i^{1,u})-\gamma_1u_i-s_1v_i=0,$$

$$\beta_2(\sigma_i^{2,v}-4\sigma_i^{1,v}+6v_i)-\gamma_2(2v_i-\sigma_i^{1,v})-s_2(2u_i-\sigma_i^{1,u})=0.$$

Again, let (u,v) be a mosaic solution of (1.1)–(1.3) satisfying Theorem 3.1. Let $(w,\bar{w})$ be a solution of (1.1)–(1.3), with $(w(0),\bar{w}(0))\in\mathcal{N}(u,v,\theta,\delta)$, and recall that $(w(0),\bar{w}(0))$ has a forward orbit which satisfies (4.12). Thus, by (4.8), we have

$$\dot{w}_i = \beta_1[2(w_i - u_i) - (\tau_i^{1,w} - \sigma_i^{1,u})] - \gamma_1(w_i - u_i) - s_1(\bar{w}_i - v_i). \tag{4.16}$$

2.1. First suppose $2\beta_1 - \gamma_1 > 0$. Then

$$\dot{w}_i = k(w_i - u_i) - [\beta_1(\tau_i^{1,w} - \sigma_i^{1,u}) - s_1(\bar{w}_i - v_i)] \tag{4.17}$$

with $k > 0$. Hence, if $w_i - u_i \le -M$, then $\tau_i^{1,w} - \sigma_i^{1,u}$ and $\bar{w}_i - v_i$ can be chosen to be sufficiently close to zero so that $\dot{w}_i \le 0$. Similarly, when $w_i - u_i \ge M$ then $\dot{w}_i \ge 0$, and so $\mathcal{N}(u, v, \theta, \delta)$ cannot be forward invariant.

2.2. Suppose $2\beta_1 - \gamma_1 < 0$, and suppose $\beta_1 > 0$ and $4\beta_1 - \gamma_1 \le -|s_1|$, as in part (2) of Lemma 4.3. Then, taking $w_i - u_i \le -M$, but $|w_j - u_j| \le M$ for all $j \ne i$, and $|\bar{w}_i - v_i| \le M$, for all i, we have

$$\dot{w}_i \ge -M(2\beta_1 - \gamma_1) - 2\beta_1 M - |s_1|M = -M[4\beta_1 - \gamma_1 + |s_1|] \ge 0,$$

since $4\beta_1 - \gamma_1 \le -|s_1|$. Also, taking $w_i - u_i \ge M$, $|w_j - u_j| \le M$ for all $j \ne i$, and $|\bar{w}_i - v_i| \le M$ for all i implies

$$\dot{w}_i \le M[4\beta_1 - \gamma_1 + |s_1|] \le 0,$$

since $4\beta_1 - \gamma_1 \le -|s_1|$.

2.3. Finally, again suppose $2\beta_1 - \gamma_1 < 0$, but now consider $\beta_1 < 0$ and $\gamma_1 \ge |s_1|$, as in part (3) of the lemma. Now, taking $w_i - u_i \le -M$, but $|w_j - u_j| \le M$ for all $j \ne i$, and $|\bar{w}_i - v_i| \le M$ for all i, gives

$$\dot{w}_i \ge -M(2\beta_1 - \gamma_1) + 2\beta_1 M - |s_1|M = M[\gamma_1 - |s_1|] \ge 0,$$

while $w_i - u_i \ge M$, $|w_j - u_j| \le M$ for all $j \ne i$, and $|\bar{w}_i - v_i| \le M$ for all i implies $\dot{w}_i \le 0$. Now, if $u_i = 1$ then $w_i \le 1 - \delta$ implies that $w_i - u_i \le -M$ and hence $\dot{w}_i \ge 0$. If $u_i = -1$ then $w_i \ge -1 + \delta$ implies that $w_i - u_i \ge M$ and hence $\dot{w}_i \le 0$. Hence, if any component w_i of w is on the boundary of $\mathcal{N}(u, v, \theta, \delta)$, then $\dot{w}_i$ points into $\mathcal{N}(u, v, \theta, \delta)$.

Now consider the equation for $\dot{\bar{w}}_i$,

$$\dot{\bar{w}}_i = \alpha[(\bar{w}_i - v_i)(6\beta_2 - 2\gamma_2) - (\tau_i^{1,\bar{w}} - \sigma_i^{1,v})(4\beta_2 - \gamma_2) + \beta_2(\tau_i^{2,\bar{w}} - \sigma_i^{2,v})$$
$$-s_2(2(w_i - u_i) - (\tau_i^{1,w} - \sigma_i^{1,u}))].$$

Suppose $0 \le 4\alpha\beta_2 \le \alpha\gamma_2$, $\alpha s_2 u_i v_i > 0$ and $u_i\sigma_i^{1,u} = -2$ as in part (5) of the lemma.

2.4. First, consider $u_i = v_i = 1$, which implies $\alpha s_2 > 0$ and $\sigma_i^{1,u} = -2$. Suppose $\bar{w}_i - v_i \le -M$, $|\bar{w}_j - v_j| \le M$ for all $j \ne i$, and $|w_i - u_i| \le M$ for all i. Then $-\alpha s_2 M \le \alpha s_2(w_i - u_i) \le 0$, since $\alpha s_2 > 0$, and also $0 \le \alpha s_2(\tau_i^{1,w} - \sigma_i^{1,u}) \le 2M$. Hence,

$$\dot{\bar{w}}_i \ge -2M\alpha(3\beta_2 - \gamma_2) + 2M\alpha(4\beta_2 - \gamma_2) - 2M\alpha\beta_2 = 0.$$

2.5. Now, consider $v_i = 1$ and $u_i = -1$ which implies $\alpha s_2 < 0$ and $\sigma_i^{1,u} = 2$, and again assume $\bar{w}_i - v_i \le -M$, $|\bar{w}_j - v_j| \le M$ for all $j \ne i$, and $|w_i - u_i| \le M$ for all i. Then $0 \ge \alpha s_2(w_i - u_i) \ge \alpha s_2 M$, and $-2\alpha s_2 \ge \alpha s_2(\tau_i^{1,w} - \sigma_i^{1,u}) \ge 0$.

Hence,

$$\dot{\bar{w}}_i \ge -2M\alpha(3\beta_2 - \gamma_2) + 2M\alpha(4\beta_2 - \gamma_2) - 2M\alpha\beta_2 + 0 = 0.$$

2.6. Similarly, if $v_i = -1$ and $u_i = \pm 1$, with $\alpha s_2 u_i v_i > 0$ and $u_i\sigma_i^{1,u} = -2$, then $\dot{\bar{w}}_i \le 0$.

Therefore, if any component $\bar{w}_i$ of $\bar{w}$ is on the boundary of $\mathcal{N}(u, v, \theta, \delta)$, then $\dot{\bar{w}}_i$ points into $\mathcal{N}(u, v, \theta, \delta)$. This, together with the result for $\dot{w}_i$ gives weak forward invariance of $\mathcal{N}(u, v, \theta, \delta)$.

We have established the lemma in the case where parts (1) and (4) hold, and also in the case where part (5) and either part (2) or part (3) hold. Clearly, the above arguments could be repeated in the other possible cases, and the result follows. □

We now identify a class of coupled mosaic solutions which are asymptotically stable. For ease of notation we will define $\bar{\beta} = \{\beta_1, \beta_2\}$, $\bar{\gamma} = \{\gamma_1, \gamma_2\}$ and $\bar{s} = \{s_1, s_2\}$.

Definition 4.4. Let $\bar{\mathcal{S}}_A(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s}) \subset \mathcal{M}_1^2$ be the set of $(u, v) \in \mathcal{M}_1^2$ such that $u_i = \pm 1$:

$$u_i[\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i] > 0$$

holds, and either

1. $v_i v_{i\pm 1} = 1$ and
$$v_i\alpha[\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - \gamma_2(2v_i - \sigma_i^{1,v}) - s_2(2u_i - \sigma_i^{1,u})] > 0$$
holds, or
2. $v_i = \pm 1$, $2v_i \neq \sigma_i^{1,v}$, $\alpha > 0$ and
$$\beta_2\left(4 - \frac{2v_i - \sigma_i^{2,v}}{2v_i - \sigma_i^{1,v}}\right) - s_2\left(\frac{2u_i - \sigma_i^{1,u}}{2v_i - \sigma_i^{1,v}}\right) > \gamma_2$$
for each $i \in \mathbb{N}$.

Theorem 4.1. *Let $(u, v) \in \bar{\mathcal{S}}_A(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$, then (u, v) is a weakly asymptotically stable mosaic solution of* (1.1)–(1.3).

Proof. Let $(u, v) \in \bar{\mathcal{S}}_A(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$, so that (u, v) satisfies the conditions of Definition 4.4. Now, since (4.9) holds, this implies that part (1.2) of Theorem 3.1 is satisfied, while case (1) of Definition 4.4 implies that part (2) of Theorem 3.1 holds, and case (2) of Definition 4.4 implies that part (4) of Theorem 3.1 is satisfied. Hence (u, v) is a mosaic solution.

Also, if $u_i = \pm 1$ and (4.9) holds, then part (1) of Lemma 4.3 is satisfied, while case (1) of Definition 4.4 implies that part (4) of Lemma 4.3 is satisfied.

For case (2) of Definition 4.4, note that $v_i = \pm 1$, $2v_i \neq \sigma_i^{1,v}$ implies that v_i has the same sign as $2v_i - \sigma_i^{1,v}$ and so, since $\alpha > 0$, we have

$$v_i\alpha[\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - \gamma_2(2v_i - \sigma_i^{1,v}) - s_2(2u_i - \sigma_i^{1,u})] > 0,$$

and hence part (4) of Lemma 4.3 holds.

So, if $(u, v) \in \bar{\mathcal{S}}_A(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$, then $\mathcal{N}(u, v, \theta, \delta)$ is weakly forward invariant for all sufficiently small $\theta, \delta > 0$. By Definition 4.2, this gives weak Lyapunov stability of (u, v).

To establish weak asymptotic stability, take M sufficiently small in the first part of the proof of Lemma 4.3, so that

$$M[|2\beta_1 - \gamma_1| + 2|\beta_1| + |s_1|] < |\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i - s_1 v_i|,$$

$$\begin{aligned} &M|\alpha|[|6\beta_2 - 2\gamma_2| + 2|4\beta_2 - \gamma_2| + 2|\beta_2| + 4|s_2|] \\ &\quad < |\alpha[\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - \gamma_2(2v_i - \sigma_i^{1,v}) - s_2(2u_i - \sigma_i^{1,u})]|. \end{aligned}$$

Now, since $u_i = \pm 1$ and (4.9) hold, we have $\dot{w}_i > 0$ when $u_i = 1$ and $(w, \bar{w}) \in \mathcal{N}(u, v, \theta, \delta)$, and $\dot{w}_i < 0$ when $u_i = -1$ and $(w, \bar{w}) \in \mathcal{N}(u, v, \theta, \delta)$, since $\dot{w}_i$ is bounded away from 0. Similarly, for v_i, since $v_i = \pm 1$ and (4.11) holds, and so weak asymptotic stability follows. □

We now identify a class of weakly Lyapunov stable coupled mosaic solutions.

Definition 4.5. Let $\bar{\mathcal{S}}_L(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s}) \subset \mathcal{M}_1^2$ be the set of $(u, v) \in \mathcal{M}_1^2$ such that one of:

1. $u_i = \pm 1$, $\beta_1(2 - \sigma_i^{1,u} u_i) - s_1 v_i u_i > \gamma_1$,
2. $u_i = \pm 1$, $\beta_1(2 - \sigma_i^{1,u} u_i) - s_1 v_i u_i = \gamma_1$, $2\beta_1 - \gamma_1 < 0$, $\beta_1 < 0$, $4\beta_1 - \gamma_1 \leq -|s_1|$,
3. $u_i = \pm 1$, $\beta_1(2 - \sigma_i^{1,u} u_i) - s_1 v_i u_i = \gamma_1$, $2\beta_1 - \gamma_1 < 0$, $\beta_1 > 0$, $\gamma_1 \geq |s_1|$, holds for each $i \in \mathbb{N}$, and either:
   - 3.1. $v_i v_{i\pm 1} = 1$ and (4.11) holds,
   - 3.2. $v_i v_{i\pm 1} = 1$, (4.10) holds, $0 \leq 4\alpha\beta_2 \leq \alpha\gamma_2$, $\alpha s_2 u_i v_i > 0$ and $u_i \sigma_i^{1,u} = -2$,
   - 3.3. $v_i = \pm 1$, $2v_i \neq \sigma_i^{1,v}$, $\alpha > 0$ and
   $$\beta_2\left(4 - \frac{2v_i - \sigma_i^{2,v}}{2v_i - \sigma_i^{1,v}}\right) - s_2\left(\frac{2u_i - \sigma_i^{1,u}}{2v_i - \sigma_i^{1,v}}\right) > \gamma_2,$$
   - 3.4. $v_i = \pm 1$, $2v_i \neq \sigma_i^{1,v}$, $\alpha > 0$, (4.10) holds, $0 \leq 4\alpha\beta_2 \leq \alpha\gamma_2$, $\alpha s_2 u_i v_i > 0$ and $u_i \sigma_i^{1,u} = -2$,
   $$\beta_2\left(4 - \frac{2v_i - \sigma_i^{2,v}}{2v_i - \sigma_i^{1,v}}\right) - s_2\left(\frac{2u_i - \sigma_i^{1,u}}{2v_i - \sigma_i^{1,v}}\right) = \gamma_2$$

   for each $i \in \mathbb{N}$.

Theorem 4.2. *Let $(u, v) \in \bar{\mathcal{S}}_L(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$ then (u, v) is a weakly Lyapunov stable mosaic solution of* (1.1)–(1.3).

Proof. Consider the cases from Definition 4.5 separately for each i. If case (1) of Definition 4.5 holds, then part (1.2) of Theorem 3.1 and part (1) of Lemma 4.3 are satisfied. If case (2) of Definition 4.5 holds, then part (1.2) of Theorem 3.1 and part (3) of Lemma 4.3 are satisfied. If case (3) of Definition 4.5 holds, then part (1.2) of Theorem 3.1 and part (2) of Lemma 4.3 are satisfied. If (u, v) satisfies case (1) of Definition 4.5, then part (3) of Theorem 3.1 and case (4) of Lemma 4.3 hold. If case (1) of Definition 4.5 is satisfied, then part (3) of Theorem 3.1 and case (5) of Lemma 4.3 hold. If (u, v) satisfies case (3) of Definition 4.5, then part (4) of Theorem 3.1 and case (4) of Lemma 4.3 hold. Finally, case (4) of Definition 4.5 implies that part (4) of Theorem 3.1 and case (5) of Lemma 4.3 hold.

Hence, if $(u, v) \in \bar{\mathcal{S}}_L(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$, then (u, v) is a mosaic solution of (1.1)–(1.3), and $\mathcal{N}(u, v, \theta, \delta)$ is weakly forward invariant for all sufficiently small $\theta, \delta > 0$, which implies weak Lyapunov stability as required. □

5. Spatial entropy

5.1. General entropy calculations

We first define the concept of spatial entropy in a general one-dimensional setting. Let $\mathcal{A}$ be a finite set of d elements and define $\mathcal{A}^{\mathbb{Z}}$ to be the set of all functions $u : \mathbb{Z} \to \mathcal{A}$. Consider any non-empty subset $\mathcal{U} \subseteq \mathcal{A}^{\mathbb{Z}}$, and assume that $\mathcal{U}$ is translation invariant, so that $S(\mathcal{U}) = \mathcal{U}$, where S is the bounded shift operator $S : \mathcal{A} \to \mathcal{A}$

$$(Su)_i = u_{i+1} \quad \forall i \in \mathbb{Z}. \tag{5.1}$$

Given any positive integer N, define the set

$$E_N := \{i \in \mathbb{Z} | 0 \leq i \leq N-1\},$$

and consider the natural projection

$$\pi_N : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{E_N},$$

which is given by restricting any $u \in \mathcal{A}^{\mathbb{Z}}$ to the finite subset of the lattice $E_N \subseteq \mathbb{Z}$. Let

$$\Gamma_N(\mathcal{U}) = \mathrm{card}(\pi_N(\mathcal{U})),$$

so that $\Gamma_N(\mathcal{U})$ counts the number of patterns which can be observed among the elements of $\mathcal{U}$, restricting observation to the subset $E_N \subseteq \mathbb{Z}$. Clearly,

$$1 \leq \Gamma_N(\mathcal{U}) \leq \mathrm{card}(\mathcal{A}^{E_N}) = d^N.$$

Note that since $\mathcal{U}$ is assumed to be translation invariant, there is no loss of generality in restricting to the coordinates $0 \leq i \leq N-1$ rather than to $c \leq i \leq N+c-1$ for some $c \neq 0$.

Definition 5.1. The *spatial entropy* of the set $\mathcal{U}$ is defined to be the limit

$$h(\mathcal{U}) := \lim_{N\to\infty} \frac{1}{N} \log \Gamma_N(\mathcal{U}).$$

Existence of this limit together with the formula

$$h(\mathcal{U}) = \inf_{N\geq 1} \frac{1}{N} \log \Gamma_N(\mathcal{U}),$$

is established in [10]; see also [1]. Note also that in general, since $\Gamma_N(\mathcal{A}^{\mathbb{Z}}) = d^N$ for any alphabet of d elements $\mathcal{A}$, then $h(\mathcal{A}^{\mathbb{Z}}) = \ln d$.

In the one-dimensional case, it is possible to calculate $h(\mathcal{U})$ explicitly when $\mathcal{U}$ belongs to a certain class of translation invariant subsets known as *Markov shifts*, or *subshifts of finite type*. These are defined as follows (see [19, p. 73]).

Let M be a $d \times d$ matrix, all of whose entries are either 1 or 0, known as a *transition matrix*, and denote the *ij*th entry of M by $M_{i,j}$. Then define the set

$$\mathcal{U}(M) = \{u \in \mathcal{A}^{\mathbb{Z}} | M_{u_i,u_{i+1}} = 1 \ \ \forall i \in \mathbb{Z}\}, \tag{5.2}$$

so that $\mathcal{U}(M)$ consists of the sequences from $\mathcal{A}^{\mathbb{Z}}$ allowed by the transition matrix M. Now note that by (5.1) and (5.2), we have $S\mathcal{U}(M) = \mathcal{U}(M)$, so that $\mathcal{U}(M)$ is translation invariant under the shift map S. Then the Markov shift for the matrix M is defined to be the map $S_d : \mathcal{U}(M) \to \mathcal{U}(M)$ defined by $S_d \equiv S|_{\mathcal{U}(M)}$.

In the case of a one-dimensional lattice, Markov shifts have been extensively studied and are well understood. In particular (see, for example [19]), we have

$$h(\mathcal{U}(M)) = \ln(\lambda), \tag{5.3}$$

where λ is the largest real positive eigenvalue of M.

The system (1.1)–(1.3) is said to exhibit *spatial chaos* at a point $(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$ in parameter space if the spatial entropy of a set of stable mosaic solutions of the system, $h(\bar{\mathcal{S}}(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s}))$, is positive. The system is said to exhibit *pattern formation* at this point if the spatial entropy is zero.

We will see below that in parameter ranges where the system exhibits pattern formation there is a fixed finite number of patterns occurring in $\bar{\mathcal{S}}(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$. Where spatial chaos occurs, $h(\bar{\mathcal{S}}(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s}))$ gives a measure of how fast $\Gamma_N(\bar{\mathcal{S}}(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s}))$ grows with N.

5.2. Entropy calculations for a spatially discrete phase transition equation

The techniques in the previous section can now be used to calculate the spatial entropy of sets of stable mosaic solutions of spatially discrete phase transition equations. This process was used in [10] to calculate the spatial entropy of the Allen–Cahn equation, and in [2] to calculate the spatial entropy for the Cahn–Hilliard equation. We define $\mathcal{F}^n = \mathcal{A}^{E_n}$ to be the set of all n-tuples $\tilde{u} = (\tilde{u}_0, \dots, \tilde{u}_{n-1})$, where $\tilde{u}_i \in \mathcal{A}$ for $i \in [0, \dots, n-1]$. Note that clearly, $\mathcal{F}^n$ has d^n elements. We now fix a non-empty subset $\mathcal{B}^n \subseteq \mathcal{F}^n$ and define a set $\tilde{\mathcal{U}}_{\mathcal{B}^n} \subseteq \mathcal{M}_1$ such that

$$\tilde{\mathcal{U}}_{\mathcal{B}^n} = \{u \in \mathcal{M}_1 | u_{i-n}, \dots, u_i, \dots, u_{i+n} \in \mathcal{B}^n \ \text{ for all } \ i \in \mathbb{Z}\}.$$

We take $\mathcal{B}^n$ to be the set of n-tuples which satisfy existence and stability conditions for the spatially discrete equation. We refer to an n-tuple $\tilde{u} \in \mathcal{B}^n$ as an *admissible n-tuple* and to $\mathcal{B}^n \subseteq \mathcal{F}^n$ as the *set of admissible n-tuples*. Thus $\tilde{\mathcal{U}}_{\mathcal{B}^n}$ is the set of mosaic solutions generated by the set of admissible n-tuples $\mathcal{B}^n$. In [10], the value of n was taken to be 3, since the Allen–Cahn equation involves values at the points u_i and $u_{i\pm1}$, whereas in [2] we took $n = 5$, since the Cahn–Hilliard equation involves values at v_i, $v_{i\pm1}$ and $v_{i\pm2}$.

To show that $\tilde{\mathcal{U}}_{\mathcal{B}^n}$ is equivalent to a Markov shift, which enables us to use (5.3), we now define an injective map which interprets a mosaic $u \in \mathcal{M}_1$ as an infinite array of $(n-1)$-tuples. Let $\mathcal{F}^{n-1} = \mathcal{A}^{E_{n-1}}$ be the set of all possible $(n-1)$-tuples $\tilde{u} = (\tilde{u}_0, \dots, \tilde{u}_{n-2})$, where again $\tilde{u}_i \in \mathcal{A}$ for $i \in [0, \dots, n-2]$. Note that $\mathcal{F}^{n-1}$ has 2^{n-1} elements. Now, given a set of admissible n-tuples $\mathcal{B}^n$, the set of *available* $(n-1)$*-tuple pairs*, $\tilde{\mathcal{B}}^{n-1} \subseteq \mathcal{F}^{n-1}$, is defined to be the set of all pairs of elements of $\mathcal{F}^{n-1}$ which overlap to give an element of $\mathcal{B}^n$, so

$$\tilde{\mathcal{B}}^{n-1} = \left\{ \hat{u}, \hat{v} \in \mathcal{F}^{n-1} \middle| \exists w = (w_0, \dots, w_{n-1}) \in \mathcal{B}^n : \begin{array}{l} \hat{u} = (w_0, \dots, w_{n-2}), \ \text{and} \\ \hat{v} = (w_1, \dots, w_{n-1}) \end{array} \right\}.$$

Define the set of *available* $(n-1)$*-tuples*, $\mathcal{B}^{n-1}$ to be those elements of $\mathcal{F}^{n-1}$ which occur as adjacent points in elements of $\mathcal{B}^n$, so

$$\mathcal{B}^{n-1} = \left\{ \hat{u} \in \hat{\mathcal{F}}^{n-1} \middle| \exists w = (w_0, \dots, w_{n-1}) \in \mathcal{B}^n : \begin{array}{l} \hat{u} = (w_0, \dots, w_{n-2}) \ \text{or} \\ \hat{u} = (w_1, \dots, w_{n-1}) \end{array} \right\}.$$

Now, define a map $\psi : \mathcal{A}^{\mathbb{Z}} \to \hat{\mathcal{F}}^{\mathbb{Z}}$ by $\psi(u)_i = \pi_{n-1}(S^i u)$ for $i \in \mathbb{Z}$, so that ψ reinterprets a mosaic $u \in \mathcal{M}_1$ as an infinite array of $(n-1)$-tuples. The map ψ is clearly injective and now $\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n})$ is a Markov chain on the set $\mathcal{B}^{n-1}$, with transition matrix M given by setting $m_{\hat{u},\hat{v}} = 1$ if and only if $\hat{u}, \hat{v} \in \mathcal{B}^{n-1}$, that is if $\hat{u}$ and $\hat{v}$ are an available $(n-1)$-tuple pair, and setting $m_{\hat{u},\hat{v}} = 0$ otherwise.

Now, $\Gamma_N(\tilde{\mathcal{U}}_{\mathcal{B}^n})$ is the number of different N-tuples $(\tilde{u}_0, \dots, \tilde{u}_{N-1})$ which occur for the elements $u \in \tilde{\mathcal{U}}_{\mathcal{B}^n}$, since $\Gamma_N(\tilde{\mathcal{U}}_{\mathcal{B}^n}) = \text{card}(\pi_N(\tilde{\mathcal{U}}_{\mathcal{B}^n}))$. Similarly, $\Gamma_{N-1}(\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n}))$ is the number of different $(N-1)$-tuples, $(\psi(u)_0, \dots, \psi(u)_{N-1})$ of elements of $\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n})$. From the definition of ψ, we see that there is a one-to-one correspondence between $\pi_N(\tilde{\mathcal{U}}_{\mathcal{B}^n})$ and $\pi_{N-1}(\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n}))$ and so for $N \geq 2$,

$$\Gamma_N(\tilde{\mathcal{U}}_{\mathcal{B}^n}) = \Gamma_{N-1}(\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n})).$$

Hence, using (5.3), it follows that

$$h(\tilde{\mathcal{U}}_{\mathcal{B}^n}) = \lim_{N\to\infty} \frac{1}{N} \ln \Gamma_N(\tilde{\mathcal{U}}_{\mathcal{B}^n}) = \lim_{N\to\infty} \left(\frac{N-1}{N}\right)\left(\frac{1}{N-1}\right) \ln \Gamma_{N-1}(\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n})) = h(\psi(\tilde{\mathcal{U}}_{\mathcal{B}^n})) = \ln \lambda,$$

where $\lambda \geq 0$ is the real part of the largest eigenvalue of the matrix M.

We will use a similar idea to calculate the spatial entropy of the coupled system. To do this, we consider solutions to a coupled system of two equations as being made up of two separate components, an infinite array of n-tuples from the first equation, and an infinite array of m-tuples from the second. Mosaic solutions of the coupled system are hence considered to be infinite arrays of vectors, which we refer to as (n, m)-tuples. We define $\mathcal{F}^n \times \mathcal{F}^m$ to be the set of all (n, m)-tuples $(\tilde{u}, \tilde{v}) = (\{\tilde{u}_0, \dots, \tilde{u}_{n-1}\}, \{\tilde{v}_0, \dots, \tilde{v}_{m-1}\})$, where $(\tilde{u}_i, \tilde{v}_i) \in \mathcal{A} \times \mathcal{A}$ for $i \in [0, \dots, n-1] \times [0, \dots, m-1]$. Clearly, $\mathcal{F}^n \times \mathcal{F}^m$ has $d^n \times d^m$ elements. We fix a non-empty subset $\mathcal{B}^n \times \mathcal{B}^m \subseteq \mathcal{F}^n \times \mathcal{F}^m$ and define a set $\tilde{\mathcal{U}}_{\mathcal{B}^n} \times \tilde{\mathcal{V}}_{\mathcal{B}^m} \subseteq \mathcal{M}_1^2$ such that

$$\tilde{\mathcal{U}}_{\mathcal{B}^n} \times \tilde{\mathcal{V}}_{\mathcal{B}^m} = \{(u, v) \in \mathcal{M}_1^2 | (\{u_{i-n}, \dots, u_{i+n}\}, \{v_{i-m}, \dots, v_{i+m}\}) \in \mathcal{B}^n \times \mathcal{B}^m \ \ \forall i \in \mathbb{Z}\}.$$

We now take $\mathcal{B}^n \times \mathcal{B}^m$ to be the set of (n, m)-tuples which satisfy existence and stability conditions for the coupled system. We refer to an (n, m)-tuple $(\tilde{u}, \tilde{v}) \in \mathcal{B}^n \times \mathcal{B}^m$ as an *admissible* (n, m)*-tuple* and to $\mathcal{B}^n \times \mathcal{B}^m \subseteq \mathcal{F}^n \times \mathcal{F}^m$ as the *set of admissible* (n, m)*-tuples*. Thus $\tilde{\mathcal{U}}_{\mathcal{B}^n} \times \tilde{\mathcal{V}}_{\mathcal{B}^m}$ is the set of mosaic solutions generated by the set of admissible (n, m)-tuples $\mathcal{B}^n \times \mathcal{B}^m$. We define the set of available $(n-1, m-1)$ pairs, and generate transition matrices M in a similar way to that given above, setting $M_{i,j} = 1$ if the ith and jth $(n-1)$-tuples overlap to form an admissible n-tuple, and also the ith and jth $(m-1)$-tuples overlap to form an admissible m-tuple. We set $M_{i,j} = 0$ otherwise. The eigenvalues of this matrix, and hence the spatial entropy of a particular region in parameter space can then be calculated numerically.

We want to calculate the spatial entropy of $\bar{\mathcal{S}}_A(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$, the set of asymptotically stable mosaic solutions to (1.1)–(1.3). Note that since the existence and stability conditions given in Definition 4.4 require $u_i = \pm 1$ and $v_i = \pm 1$, we will always assume that $\mathcal{A} = \{-1, 1\}$, so that $d = 2$.

The conditions in Definition 4.4 give rise to inequalities which define the boundaries of the parameter regions for the coupled system. For the Allen–Cahn component, from (4.9), we obtain six inequalities

$$0 > \pm s_1 + \gamma_1, \qquad 2\beta_1 > \pm s_1 + \gamma_1, \qquad 4\beta_1 > \pm s_1 + \gamma_1, \tag{5.4}$$

while (4.11) gives 33 inequalities for the Cahn–Hilliard component which form the boundaries of parameter regions. These are

$$\begin{aligned}
&0 > \pm s_2, \quad 4\beta_2 > \pm 2s_2 + \gamma_2, \quad 4\beta_2 > \pm s_2 + \gamma_2, \quad 4\beta_2 > \gamma_2, \quad 8\beta_2 > \pm s_2 + \gamma_2,\\
&-\beta_2 > \pm 2s_2, \quad -\beta_2 > \pm s_2, \quad -\beta_2 > 0, \quad 3\beta_2 > \pm 2s_2 + \gamma_2, \quad 3\beta_2 > \pm s_2 + \gamma_2,\\
&3\beta_2 > \gamma_2, \quad 7\beta_2 > \pm 2s_2 + 2\gamma_2, \quad 7\beta_2 > \pm s_2 + 2\gamma_2, \quad 7\beta_2 > 2\gamma_2, \quad -2\beta_2 > \pm s_2,\\
&2\beta_2 > \pm 2s_2 + \gamma_2, \quad 2\beta_2 > \pm s_2 + \gamma_2, \quad 2\beta_2 > \gamma_2, \quad 6\beta_2 > \pm s_2 + 2\gamma_2.
\end{aligned}$$

From these two sets of inequalities, it is clear that the set of all possible parameter regions for the coupled system is both large and complex. We treat parameters s_1 and s_2 as scaling values and hence assume that $s_1 = \pm 1$ and $s_2 = \pm 1$. With this assumption, we find that there are 17 possible (γ_1, β_1) regions for the u component, and 265 (γ_2, β_2) regions for the v component. These are shown in Fig. 1. We have not labeled each separate v component as the number of regions makes this difficult to see.

The regions defined in Fig. 1 clearly include one of the more physically relevant parameter regions, $\beta_1 < 0$ and $\alpha \cdot \beta_2 < 0$ that was considered in [8].

Note that some of the region boundaries are shown as solid, and some as dotted lines. The solid lines in the left-hand graph of Fig. 1 represent boundaries which are independent of v_i and the dotted lines show regions in which the admissibility of a particular mosaic solution depends on the value of v_i. For example, the first inequality in (5.4) with $s_1 = \pm 1$, becomes $\gamma_1 < -1$ if $s_1 v_i = 1$, and $\gamma_1 < 1$ if $s_1 v_i = -1$. Clearly, if $\gamma_1 < -1$, both these inequalities are satisfied, so admissibility of solutions in regions to the left of the solid line $\gamma_1 = -1$ does not depend on v_i in this case. If $-1 < \gamma_1 < 1$, then solutions may be admissible providing $s_1 v_i = -1$ and this is indicated by

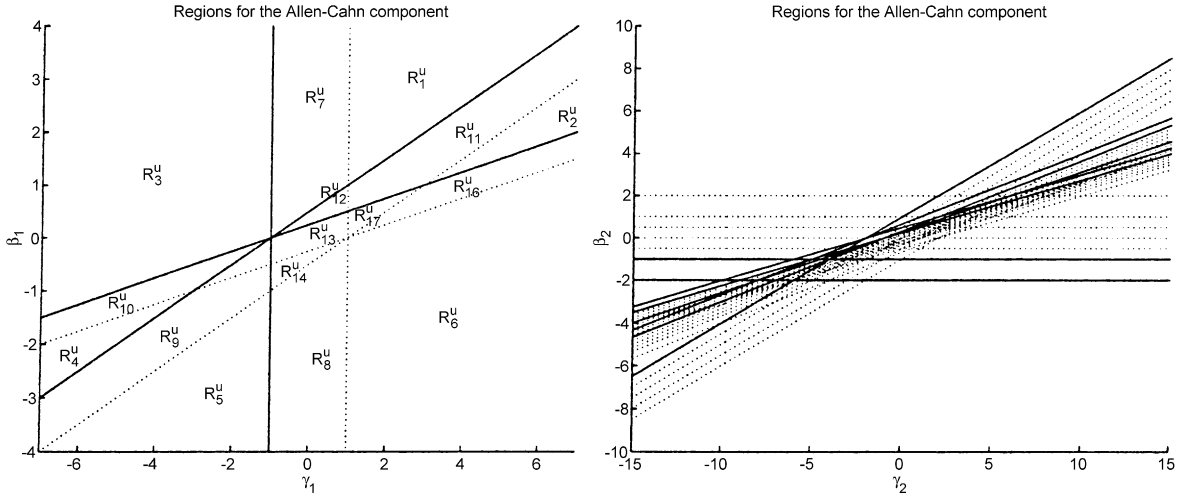

Fig. 1. Spatial entropy (γ_1, β_1) regions for the Allen–Cahn component and (γ_2, β_2) regions for the Cahn–Hilliard component.

the dotted line $\gamma_1 = 1$. To the right of the line $\gamma_1 = 1$, this inequality cannot be satisfied. Similarly, in the right-hand graph of Fig. 1, solid lines represent boundaries which are independent of u_i and $u_{i\pm1}$, and the dotted lines show regions in which the admissibility of a particular mosaic solution depends on the value of $u_{i\pm1}$.

Clearly, in the coupled case the components are dependent on each other, and hence there are $2\times17\times265$ possible $(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$ regions, in other words, 9010 potential entropy calculations! We therefore start with a simpler and less time-consuming problem, by considering the spatial entropy of an essentially decoupled version of (1.1)–(1.3).

5.3. Spatial entropy for the decoupled system

We make the following definition, which arises naturally from the set of inequalities above.

Definition 5.2. Let $\bar{\mathcal{S}}_A^{\mathrm{dec}}(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s}) \subset \mathcal{M}_1^2$ be the set of $(u, v) \in \mathcal{M}_1^2$ such that $u_i = \pm1$, $v_i = \pm1$,

$$u_i[\beta_1(2u_i - \sigma_i^{1,u}) - \gamma_1 u_i] > |s_1|, \tag{5.6}$$

$$v_i[\alpha(\beta_2(\sigma_i^{2,v} - 4\sigma_i^{1,v} + 6v_i) - \gamma_2(2v_i - \sigma_i^{1,v}))] > 4\alpha|s_2| \tag{5.7}$$

for each $i \in \mathbb{N}$.

This definition essentially decouples the system, since the Allen–Cahn component now only depends on u_i and the Cahn–Hilliard component only depends on v_i. It is clearly much simpler to calculate the spatial entropy of the set $\bar{\mathcal{S}}_A^{\mathrm{dec}}(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$ than $\bar{\mathcal{S}}_A(\alpha, \bar{\beta}, \bar{\gamma}, \bar{s})$ as the decoupling means that we can calculate the entropy of the two components separately. By taking the absolute value of s_1 and s_2, we obtain solutions which exist independently of the value of v_i, and of u_i and $u_{i\pm1}$ respectively.

5.3.1. Entropy for the Allen–Cahn component

To calculate the spatial entropy for the decoupled Allen–Cahn component, we use the method given in Section 5.2 with $n = 3$ since the Allen–Cahn equation involves values at the points u_i and $u_{i\pm1}$. In each region of parameter space (β_1, γ_1), we check which of the $2^3 = 8$ three-tuples are admissible, and hence generate a 4×4 transition matrix

Table 1
Possible 2-tuples and general form of the transition matrix for the u component in the decoupled case

Label	Two-tuple
a	(1 1)
b	(1 −1)
c	(−1 1)
d	(−1 −1)

$$\begin{pmatrix} m_{a,a} & m_{a,b} & 0 & 0 \\ 0 & 0 & m_{b,c} & m_{b,d} \\ m_{c,a} & m_{c,b} & 0 & 0 \\ 0 & 0 & m_{d,c} & m_{d,d} \end{pmatrix}$$

M^u with $M^u_{i,j} = 1$ if the ith and jth 2-tuples overlap to form an admissible three-tuple, and $M^u_{i,j} = 0$ otherwise. The eigenvalues of this matrix, and hence the spatial entropy of the region can then be calculated numerically.

Each of the different parameter regions in Fig. 1 results in a different 4×4 transition matrix M^u. However, these matrices are all very sparse and have only a few non-zero eigenvalues, and moreover the transition matrices M^u of several of the regions have the same eigenvalues and eigenvectors, despite the matrices themselves being different.

To understand this, consider that the transition matrix M^u can be thought of as a map, with $M^u_{i,j} = 1$ if the ith and jth 2-tuples overlap to form an available three-tuple. Given two such 2-tuples, suppose we now want to extend the resulting three-tuple into a four-tuple, as the next step to constructing a mosaic solution. To do so, we require a two-tuple, which overlaps with the jth two-tuple to form a three-tuple. Such a two-tuple exists if and only if $M^u_{j,k} = 1$ for some k. If $M^u_{j,k} = 0$ for all k then even though $M^u_{i,j} = 1$ no mosaic solution will exist that contains the corresponding admissible three-tuple.

Hence, some of the entries in the transition matrices can only be used to form mosaic solutions on finite subsets of $\mathbb{Z}$, and since we are considering solutions which are projections of an infinite mosaic solution onto a finite subset of the lattice, these entries can be deleted to give a condensed transition matrix $\tilde{M}^u$. Note that M^u and $\tilde{M}^u$ have the same eigenvalues, as the rows and columns of M^u deleted to form $\tilde{M}^u$ always fall in the null-space of M^u.

It is possible to use $\tilde{M}^u$ to construct words $\pi_N(u) \in \mathcal{A}^{E_N}$ of any Markov chain u for $N \geq 2$, and we can calculate $\Gamma_N(\mathcal{U})$ using the following theorem.

Theorem 5.1. *Denote the* i, j*-entry of* $(\tilde{M}^u)^k$ *by* $(\tilde{M}^u)^k_{i,j}$. *Then there are* $(\tilde{M}^u)^k_{i,j}$ *allowable words of length* $k+1$ *starting at i and ending at j, i.e.* $(\tilde{M}^u)^k_{i,j}$ *words of the form* $is_1, \dots, s_{k-1}j$.

Proof. See [19, Lemma 2.2, p. 24]. □

Clearly, the total number of words of length $k+1$ is given by summing the elements of $(\tilde{M}^u)^k$ and hence

$$\Gamma_N(\mathcal{U}) = \sum_{i,j=1}^{d} (\tilde{M}^u)^k_{i,j}.$$

The four 2-tuples which can appear in mosaic solutions, and the general form of the condensed transition matrix $\tilde{M}^u$, indicating the possible non-zero entries, are given in Table 1.

We find that there are six different condensed transition matrices for the Allen–Cahn component in the decoupled system, which we label $\tilde{M}^u_i$ for $i = 1, \dots, 6$. The 3-tuples generated by the non-zero entries of each matrix, and some simple periodic solutions arising for each condensed transition matrix are given in Table 2.

The regions of (β_1, γ_1) parameter space in which each condensed transition matrix occurs are given in Fig. 2.

Note that the dashed line shown on the graph in Fig. 2 denotes the boundary between regions in which either no stable mosaic solution exists or pattern formation occurs, and regions exhibiting spatial chaos.

Table 3 gives the spatial entropy of each of the transition matrices $\tilde{M}^u_i$ in the order of increasing spatial entropy. The columns headed $\Gamma_N(S^{\text{dec}}_A)$ give the number of different N-tuples which arise as projections of mosaic solutions

Table 2
Decoupled system: admissible 3-tuples and simple periodic solutions arising for each condensed transition matrix for the decoupled u component

Three-tuple	$\widetilde{M}_1^u$	$\widetilde{M}_2^u$	$\widetilde{M}_3^u$	$\widetilde{M}_4^u$	$\widetilde{M}_5^u$	$\widetilde{M}_6^u$
(1 1 1)		•		•		•
(1 1 −1)				•	•	•
(1 −1 1)			•		•	•
(1 −1 −1)				•	•	•
(−1 1 1)				•	•	•
(−1 1 −1)			•		•	•
(−1 −1 1)					•	•
(−1 −1 −1)		•		•		•

Mosaic Solution	$\widetilde{M}_1^u$	$\widetilde{M}_2^u$	$\widetilde{M}_3^u$	$\widetilde{M}_4^u$	$\widetilde{M}_5^u$	$\widetilde{M}_6^u$
... − 1 ...		•		•		•
... 1 ...		•		•		•
... − 11 ...			•		•	•
... − 1 − 11 ...						•
... − 111 ...						•
... − 1 − 111 ...						•

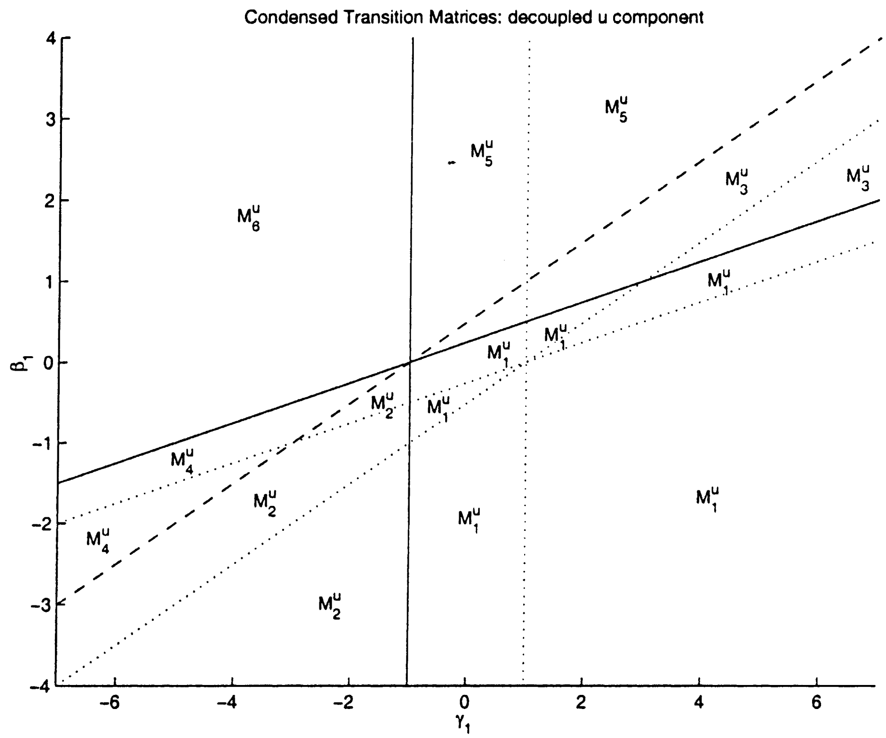

Fig. 2. Parameter value regions.

Table 3
Decoupled system: spatial entropy and number of mosaic solutions in a subset of $\mathbb{Z}$ for each u component condensed transition matrix

Case	Entropy	$\Gamma_3(S_A^{\rm dec})$	$\Gamma_{50}(S_A^{\rm dec})$	$\Gamma_{1000}(S_A^{\rm dec})$
$\tilde{M}_1^u$	ln 0	0	0	0
$\tilde{M}_2^u$	ln 1 = 0	2	2	2
$\tilde{M}_3^u$	ln 1 = 0	2	2	2
$\tilde{M}_4^u$	ln 1.6180	6	4.0730×10^{10}	1.4066×10^{209}
$\tilde{M}_5^u$	ln 1.6180	6	4.0730×10^{10}	1.4066×10^{209}
$\tilde{M}_6^u$	ln 2	8	1.1259×10^{15}	1.0715×10^{301}

onto a subset of $\mathbb{Z}$ of length j. Note from above that $\Gamma_3(S_A^{\text{dec}})$ is the same as the number of non-zero entries of $\tilde{M}$, while for $N > 3$ this is given by summing all the entries of $\tilde{M}^{N-2}$. Note that $\tilde{M}_1^u$ has no non-zero entries. In this case S_A^{dec} is empty, that is there are no Lyapunov stable mosaic solutions in S_A^{dec} for the corresponding parameter values, and accordingly the spatial entropy is undefined. The condensed transition matrices $\tilde{M}_2^u$ and $\tilde{M}_3^u$ do have non-zero elements, but in each case the spatial entropy is zero and there is a fixed number of spatially periodic Lyapunov stable solutions in S_A^{dec} for each case. The remaining cases $\tilde{M}_i^u$, $i \geq 4$ are more interesting with positive entropy and spatial chaos; as illustrated by the rapid growth of $\Gamma_N(S_A^{\text{dec}})$ as N is increased.

Note that $\tilde{M}_4^u$ and $\tilde{M}_5^u$ are listed separately, even though the entries in this table for the two matrices are identical. Despite having the same entropy value and the same number of possible mosaic solutions on a subset of $\mathbb{Z}$, the two transition matrices are different, and therefore will give different solutions. For example, as can be seen from Table 2, the class of solutions given by $\tilde{M}_4^u$ includes the constant solutions $u_i = -1$ and $u_i = 1$ for all $i \in \mathbb{Z}$ but not the alternating solution $u_i = \pm 1$ for all $i \in \mathbb{Z}$, whereas $\tilde{M}_5^u$ gives the alternating solution but not the constant solutions.

5.4. *Entropy for the Cahn–Hilliard component*

We now calculate the spatial entropy for the v-component in the decoupled system. This is done in the same way as for the u component, but since the Cahn–Hilliard equations uses the values of v_i, $v_{i\pm1}$ and $v_{i\pm2}$, we now take $n = 5$ in the definitions in Section 5.2. Each of the different parameter regions in Fig. 1 therefore results in a different 16×16 transition matrix M^v, but, as above, several of the regions have the same eigenvalues and eigenvectors, despite the matrices themselves being different. We also find that for the same reasons given for the Allen–Cahn component, only certain entries in the transition matrix form a five-tuple combination which is not only admissible, but which can also be used to form part of a translation invariant mosaic solution. Thus, it is sufficient to display a matrix $\tilde{M}^v$ which we again refer to as a condensed transition matrix. As before, M^v and $\tilde{M}^v$ have the same eigenvalues, as the rows and columns of M deleted to form $\tilde{M}^v$ always fall in the null-space of M^v.

The 16 four-tuples, and the general form of the condensed transition matrix $\tilde{M}^v$ are given in Table 4. Note that the general form of the condensed transition matrix is given in a concise form which just shows the number of each row of the matrix and the possible non-zero entries for that row.

Table 4
Possible 4-tuples and the concise general form of the transition matrix for the v component in the decoupled case, showing only the possible non-zero entries in each row

Label	Four-tuple			
1	(1	1	1	1)
2	(1	1	1	−1)
3	(1	1	−1	1)
4	(1	1	−1	−1)
5	(1	−1	1	1)
6	(1	−1	1	−1)
7	(1	−1	−1	1)
8	(1	−1	−1	−1)
9	(−1	1	1	1)
10	(−1	1	1	−1)
11	(−1	1	−1	1)
12	(−1	1	−1	−1)
13	(−1	−1	1	1)
14	(−1	−1	1	−1)
15	(−1	−1	−1	1)
16	(−1	−1	−1	−1)

1	2	
2	3	4
3	5	6
4	7	8
5	9	10
6	11	12
7	13	14
8	15	16
9	1	2
10	3	4
11	5	6
12	7	8
13	9	10
14	11	12
15	13	14
16	15	

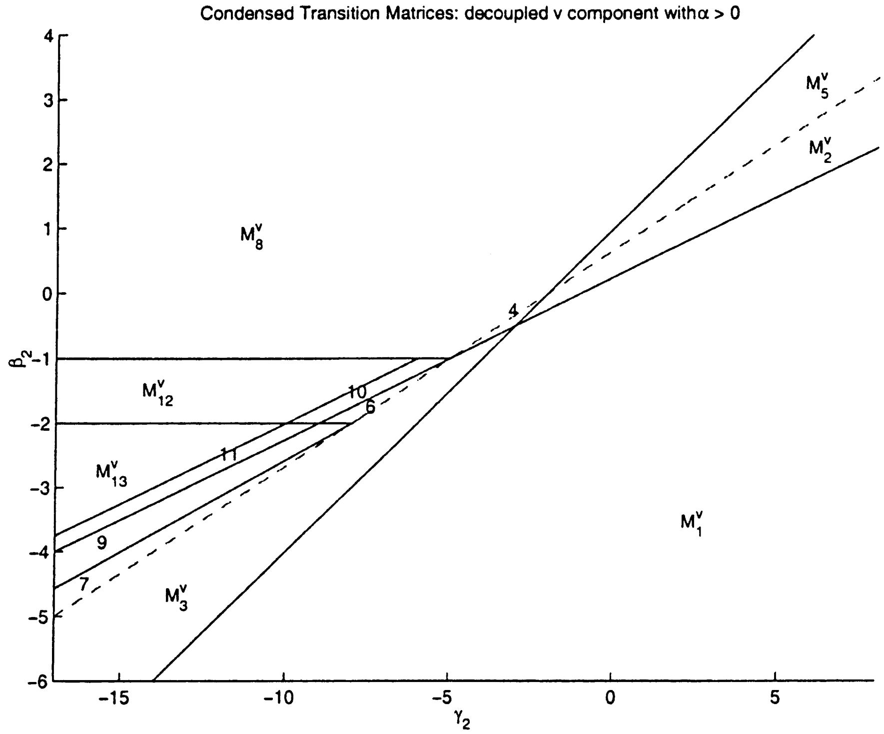

Fig. 3. Entropy regions: v component with $\alpha > 0$, decoupled case.

Fig. 3 shows the various entropy regions for the v component in the decoupled case with $\alpha > 0$, and Fig. 4 shows the entropy regions with $\alpha < 0$. As before the dotted lines indicate the border between pattern formation and spatial chaos (Tables 5 and 6).

Table 6 gives the spatial entropy and the number of possible mosaic solutions on a finite subset of $\mathbb{Z}$ for the 13 condensed transition matrices $\tilde{M}_i^v$ for $i = 1, \ldots, 13$.

Table 5
Simple periodic solutions arising for the different transition matrices $\alpha > 0$

Mosaic solution	$\tilde{M}_1^v$	$\tilde{M}_2^v$	$\tilde{M}_3^v$	$\tilde{M}_4^v$	$\tilde{M}_5^v$	$\tilde{M}_6^v$	$\tilde{M}_7^v$	$\tilde{M}_8^v$	$\tilde{M}_9^v$	$\tilde{M}_{10}^v$	$\tilde{M}_{11}^v$	$\tilde{M}_{12}^v$	$\tilde{M}_{13}^v$
$\ldots -1 \ldots$													
$\ldots 1 \ldots$													
$\ldots -11 \ldots$		•		•	•		•		•		•	•	•
$\ldots -1-11 \ldots$					•	•	•	•	•	•	•	•	•
$\ldots -111 \ldots$					•	•	•	•	•	•	•	•	•
$\ldots -1-1-11 \ldots$												•	•
$\ldots -1-111 \ldots$			•	•		•	•	•		•	•	•	•
$\ldots -1111 \ldots$												•	
$\ldots -1-1-1-11 \ldots$													•
$\ldots -1-1-111 \ldots$						•	•		•	•	•	•	•
$\ldots -1-1111 \ldots$						•	•		•	•	•	•	•
$\ldots -11111 \ldots$													•
$\ldots -1-11-11 \ldots$										•	•	•	•
$\ldots -11-111 \ldots$										•	•	•	•
$\ldots -1-1-1111 \ldots$						•	•		•	•	•	•	•
$\ldots -1-1-1-11111 \ldots$							•		•		•		•

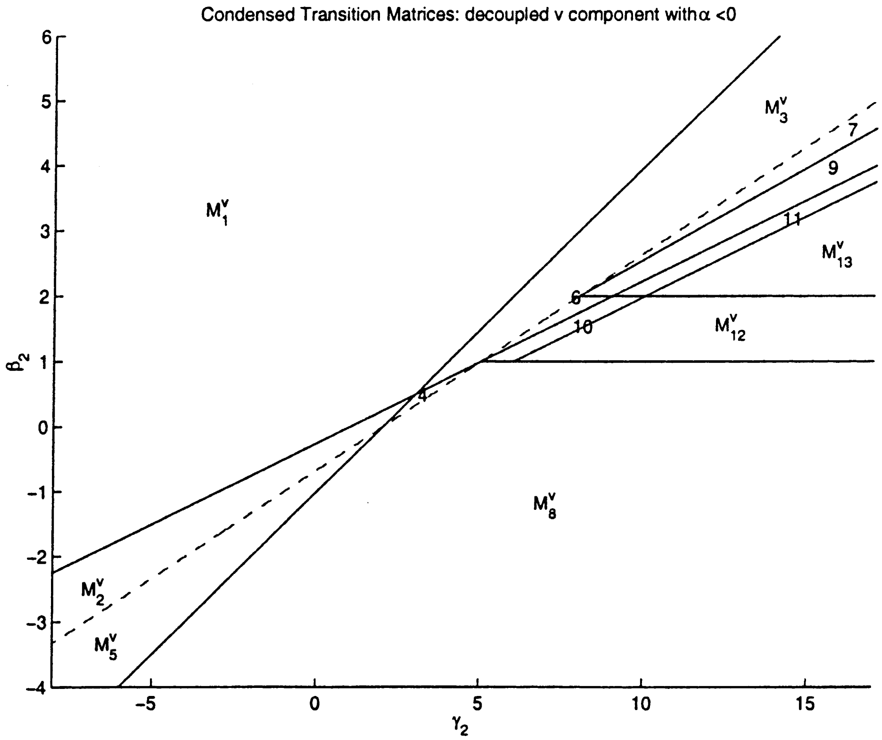

Fig. 4. Entropy regions: v component with $\alpha < 0$, decoupled case.

5.5. *Spatial entropy for the coupled system*

To calculate the spatial entropy for the coupled system (1.1)–(1.3) we use the technique explained in Section 5.1, with $n = 3$ and $m = 5$, so that we are considering solutions to the coupled system as infinite arrays of three 5-tuples. Transition matrices are then constructed by considering the availability and admissibility of two 4-tuples. We again

Table 6
Decoupled system: spatial entropy and number of mosaic solutions in a subset of $\mathbb{Z}$ for condensed transition matrices: v component

Case	Entropy	$\Gamma_5(S_A^{\rm dec})$	$\Gamma_{50}(S_A^{\rm dec})$	$\Gamma_{1000}(S_A^{\rm dec})$
$\tilde{M}_1^v$	ln 0	0	0	0
$\tilde{M}_2^v$	ln 1 = 0	2	2	2
$\tilde{M}_3^v$	ln 1 = 0	4	4	4
$\tilde{M}_4^v$	ln 1 = 0	6	6	6
$\tilde{M}_5^v$	ln 1.4656	12	357910366	1.8212×10^{166}
$\tilde{M}_6^v$	ln 1.5552	20	8.4872×10^{9}	1.3253×10^{192}
$\tilde{M}_7^v$	ln 1.5552	20	8.4872×10^{9}	1.3253×10^{192}
$\tilde{M}_8^v$	ln 1.6180	16	4.0730×10^{10}	1.4066×10^{209}
$\tilde{M}_9^v$	ln 1.6180	24	5.8945×10^{10}	2.0357×10^{209}
$\tilde{M}_{10}^v$	ln 1.6663	22	2.0445×10^{11}	9.4753×10^{221}
$\tilde{M}_{11}^v$	ln 1.7024	26	6.2123×10^{11}	1.9843×10^{231}
$\tilde{M}_{12}^v$	ln 1.8393	26	2.1124×10^{13}	5.5177×10^{264}
$\tilde{M}_{13}^v$	ln 1.9276	30	2.0162×10^{14}	1.1547×10^{285}

find that not all entries in the transition matrix will give three 5-tuples which can be used to form mosaic solutions over the whole of $\mathbb{Z}$, and so it is sufficient to consider $\tilde{M}$, a 64×64 condensed transition matrix. The two 4-tuples and the general form of the condensed transition matrix for the coupled system is given in Table 7. We have again given the general form of the transition matrix in a shortened form, just showing the possible non-zero entries for each row of the matrix.

Note the two-four-tuple pairs (1, 1), (16, 16), (49, 49) and (64, 64) do not appear in the general matrix as possible non-zero entries. In fact, these always have $M_{i,j} = 0$, since they result in constant solutions in both components simultaneously. In fact, if we have a constant solution in the v component, then

$$\dot{v}_i = -v_i \alpha [s_2(2u_i - \sigma_i^{1,u})],$$

and so for admissibility, we require

$$v_i \alpha [s_2(2u_i - \sigma_i^{1,u})] < 0.$$

This cannot occur when $u_i = 1$ for all i or $u_i = -1$ for all i, since this gives $2u_i - \sigma_i^{1,u} = 0$. Hence u and v cannot both be constant.

We will not attempt to give explicit results for the spatial entropy of all 9010 possible coupled parameter regions! Instead, we will classify the parameter space for the coupled system, by using the spatial entropy of the solutions in each region to determine the extent to which the equations effectively act in an uncoupled manner, the extent to which a single equation in the system drives the system, and the extent to which the system is highly coupled and is not driven by a single equation. Hence, we will present plots of parameter space which may be used to determine the type of coupling for a given set of parameter values.

5.5.1. No mosaic solutions

If (β_1, γ_1) are in the region marked as $u_{\rm A}$ in Fig. 5, then no mosaic solutions exist for any values of $(\alpha, \beta_2, \gamma_2, s_2)$. This is because (β_1, γ_1) parameter values in this region, with $s_1 = \pm 1$ cannot satisfy any of the six inequalities in (5.4), and hence all the entries in each transition matrix for this region are zero.

Now, if (β_1, γ_1) are in the parameter region labeled $u_{\rm B}$, then the transition matrices generated for several values of $(\alpha, \beta_2, \gamma_2, s_2)$ do have some non-zero entries. However, the condensed transition matrices obtained from these, which indicate three 5-tuples which are not only admissible, but which can be used to from part of an infinite mosaic solution, have no non-zero entries. Hence all eigenvalues of the condensed transition matrix are zero for all values of $(\alpha, \beta_2, \gamma_2, s_2)$, and again no mosaic solutions exist.

In both cases, the coupled solution is dominated by the Allen–Cahn component, since it only depends on the Allen–Cahn parameters. We can predict the behavior of the coupled solution just by knowing the values of (β_1, γ_1, s_1), the values of $(\alpha, \beta_2, \gamma_2, s_2)$ are irrelevant in these cases.

If we take (β_2, γ_2) in the region labeled v_A in Fig. 6, a similar situation occurs. Again, while the full transition matrix has non-zero entries, the condensed transition matrix contains only zero entries, and hence the eigenvalues of the condensed transition matrix are all 0 for any (β_1, γ_1, s_1) parameter region R^u.

In this case, it is the Cahn–Hilliard component of the coupled solution which dominates, and we can predict that there will be no mosaic solutions if (β_2, γ_2) are in this parameter region, regardless of the values of (β_1, γ_1, s_1). Note that the diagram in Fig. 6 is for $\alpha > 0$. The diagram for $\alpha < 0$ will be the mirror-image of this, in the same way as the diagrams in Figs. 3 and 4 are mirror-images of each other.

5.5.2. Pattern formation

When (β_1, γ_1) are in the parameter region labeled $u_{\rm C}$ the maximum entropy value we can obtain is ln 1, indicating pattern formation, for any value of $(\alpha, \beta_2, \gamma_2, s_2)$. Although in this case the coupled solution is not entirely

Table 7
Two 4-tuples and the general condensed transition matrix for the coupled case

Label	Two-four-tuple					
1	1	1	1	1	1	1
2	1	1	1	1	1	-1
3	1	1	1	1	-1	1
4	1	1	1	1	-1	-1
5	1	1	1	-1	1	1
6	1	1	1	-1	1	-1
7	1	1	1	-1	-1	1
8	1	1	1	-1	-1	-1
9	1	1	-1	1	1	1
10	1	1	-1	1	1	-1
11	1	1	-1	1	-1	1
12	1	1	-1	1	-1	-1
13	1	1	-1	-1	1	1
14	1	1	-1	-1	1	-1
15	1	1	-1	-1	-1	1
16	1	1	-1	-1	-1	-1
17	1	-1	1	1	1	1
18	1	-1	1	1	1	-1
19	1	-1	1	1	-1	1
20	1	-1	1	1	-1	-1
21	1	-1	1	-1	1	1
22	1	-1	1	-1	1	-1
23	1	-1	1	-1	-1	1
24	1	-1	1	-1	-1	-1
25	1	-1	-1	1	1	-1
27	1	-1	-1	1	1	1
26	1	-1	-1	1	1	-1
27	1	-1	-1	1	-1	1
28	1	-1	-1	1	-1	-1
29	1	-1	-1	-1	1	1
30	1	-1	-1	-1	1	-1
31	1	-1	-1	-1	-1	1
32	1	-1	-1	-1	-1	-1
33	-1	1	1	1	1	1
34	-1	1	1	1	1	-1
35	-1	1	1	1	-1	1
36	-1	1	1	1	-1	-1
37	-1	1	1	-1	1	1
38	-1	1	1	-1	1	-1
39	-1	1	1	-1	-1	1
40	-1	1	1	-1	-1	-1
41	-1	1	-1	1	1	1
42	-1	1	-1	1	1	-1
43	-1	1	-1	1	-1	1
44	-1	1	-1	1	-1	-1
45	-1	1	-1	-1	1	1
46	-1	1	-1	-1	1	-1
47	-1	1	-1	-1	-1	1
48	-1	1	-1	-1	-1	-1
49	-1	-1	1	1	1	1
50	-1	-1	1	1	1	-1
51	-1	-1	1	1	-1	1
52	-1	-1	1	1	-1	-1
53	-1	-1	1	-1	1	1
54	-1	-1	1	-1	1	-1
55	-1	-1	1	-1	-1	1
56	-1	-1	1	-1	-1	-1
57	-1	-1	-1	1	1	1
58	-1	-1	-1	1	1	-1
59	-1	-1	-1	1	-1	1
60	-1	-1	-1	1	-1	-1
61	-1	-1	-1	-1	1	1
62	-1	-1	-1	-1	1	-1
63	-1	-1	-1	-1	-1	1
64	-1	-1	-1	-1	-1	-1

1		2	17	18
2	3	4	19	20
3	5	6	21	22
4	7	8	23	24
5	9	10	25	26
6	11	12	27	28
7	13	14	29	30
8	15	16	31	32
9	1	2	17	18
10	3	4	19	20
11	5	6	21	22
12	7	8	23	24
13	9	10	25	26
14	11	12	27	28
15	13	14	29	30
16	15		31	32
17	33	34	49	50
18	35	36	51	52
19	37	38	53	54
20	39	40	55	56
21	41	42	57	58
22	43	44	59	60
23	45	46	61	62
24	47	48	63	64
25	33	34	49	50
26	35	36	51	52
27	37	38	53	54
28	39	40	55	56
29	41	42	57	58
30	43	44	59	60
31	45	46	61	62
32	47	48	63	64
33	1	2	17	18
34	3	4	19	22
35	5	6	21	22
36	7	8	23	24
37	9	10	25	26
38	11	12	27	28
39	13	14	29	30
40	15	16	31	32
41	1	2	17	18
42	3	4	19	20
43	5	6	21	22
44	7	8	23	24
45	9	10	25	26
46	11	12	27	28
47	13	14	29	30
48	15	16	31	32
49	33	34		50
50	35	36	51	52
51	37	38	53	54
52	39	40	55	56
53	41	42	57	58
54	43	44	59	60
55	45	46	61	62
56	47	48	63	64
57	33	34	49	50
58	35	36	51	52
59	37	38	53	54
60	39	40	55	56
61	41	42	57	58
62	43	44	59	60
63	45	46	61	62
64	47	48	63	

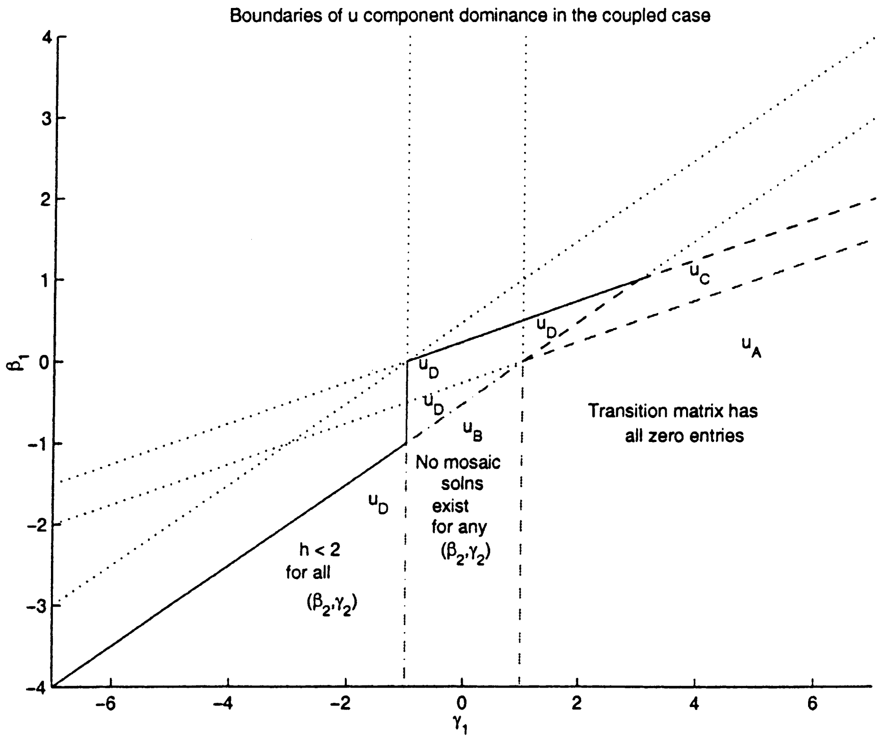

Fig. 5. Boundaries of component dominance in the coupled case: Allen–Cahn component.

independent of the Cahn–Hilliard component, as the values of $(\alpha, \beta_2, \gamma_2, s_2)$ may affect whether solutions exist at all, the coupled solution is still dominated by the Allen–Cahn component since, even if the decoupled system shows that spatially chaotic solutions exist for particular values of $(\alpha, \beta_2, \gamma_2, s_2)$, if we take (β_1, γ_1) in this region of parameter space we can only expect to find at most pattern formation in the coupled solution.

A similar situation occurs when (β_2, γ_2) are in the parameter region marked as v_B in Fig. 6. Many of the Allen–Cahn parameter regions have the potential for spatially chaotic solutions, which is clear from considering the decoupled system. However, the decoupled analysis shows solutions which exist independently of the v component, and when (β_2, γ_2) are in this parameter range, the Cahn–Hilliard component dominates the coupled solution, producing a maximum entropy value of ln 1.

5.5.3. Spatial chaos

We now consider parameter regions in which an entropy value of greater than ln 1 is possible, indicating the possibility of spatially chaotic solutions occurring in one or both components.

If we take (β_1, γ_1) in the region labeled u_D in Fig. 5 and (β_2, γ_2) in the region shown as v_C in Fig. 6, then the entropy for the coupled system is ln 1.46557, which indicates spatial chaos. In fact, in this case we actually only have the potential for chaotic behavior in the Cahn–Hilliard component, as the only entries in the condensed transition matrix in this case are (i, j), with both $i, j < 16$, giving the constant solution $u_i = 1$ for all $i \in \mathbb{Z}$, and (k, l), with $49 < k, l < 64$, giving the constant solution $u_i = -1$ for all $i \in \mathbb{Z}$.

Regions marked u_D in Fig. 5, and region v_C in Fig. 6, lead to entropy values $h \leq \ln 2$ indicating at most two entries in each row of the matrix.

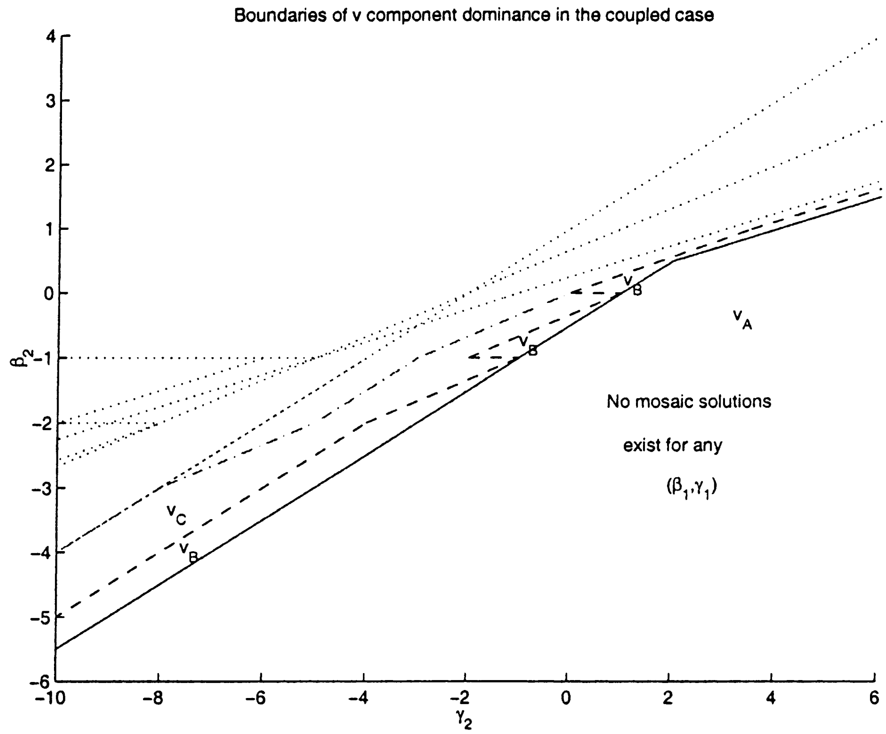

Fig. 6. Boundaries of component dominance in the coupled case: Cahn–Hilliard component.

The highest entropy value obtained in the coupled case is $\ln 3.8876$. Note that this value is close to $\ln(2 \times 2)$, which is the maximum possible number of entries in each row and column of the condensed transition matrix. The entropy is actually less than $\ln 4$, since four rows have only three elements (pairs giving constant solutions in both components do not appear, as noted above). As could be expected, this value is obtained when (β_1, γ_1) are in the region labeled M_6^u in Fig. 2 and (β_2, γ_2) are in the region shown as M_{13}^v in Figs. 3 and 4, which are the regions of highest entropy in the decoupled cases.

6. Numerical simulations

We illustrate results from previous sections with numerical simulations of a coupled discrete coupled Allen–Cahn/Cahn–Hilliard system. The differential inclusion (1.1)–(1.3) is difficult to simulate directly due to the set-valued nonlinearities and obstacles, so instead, we simulate the problem considered in Section 2, where we replace f_1 and f_2 by f_1^ε and f_2^ε as defined by (2.2) and (2.3). We then take small values of ε, typically $\varepsilon = 5 \times 10^{-4}$. All computations were performed on one-dimensional lattices $[-1, 1]^{\mathbb{Z}}$ with periodic boundary conditions $u_i = u_{i+N}$ and $v_i = v_{i+N}$, and computations were carried out using the Implicit Trapezoidal Rule. As this method requires a Newton iteration to be performed at each time step, we approximate the first derivatives of the functions f_1^ε and f_2^ε by

$$g_1^\varepsilon(u, v) = \begin{cases} \gamma_1 & \text{if } |u| < 1, \\ 0 & \text{if } |u| > 1, \end{cases}$$

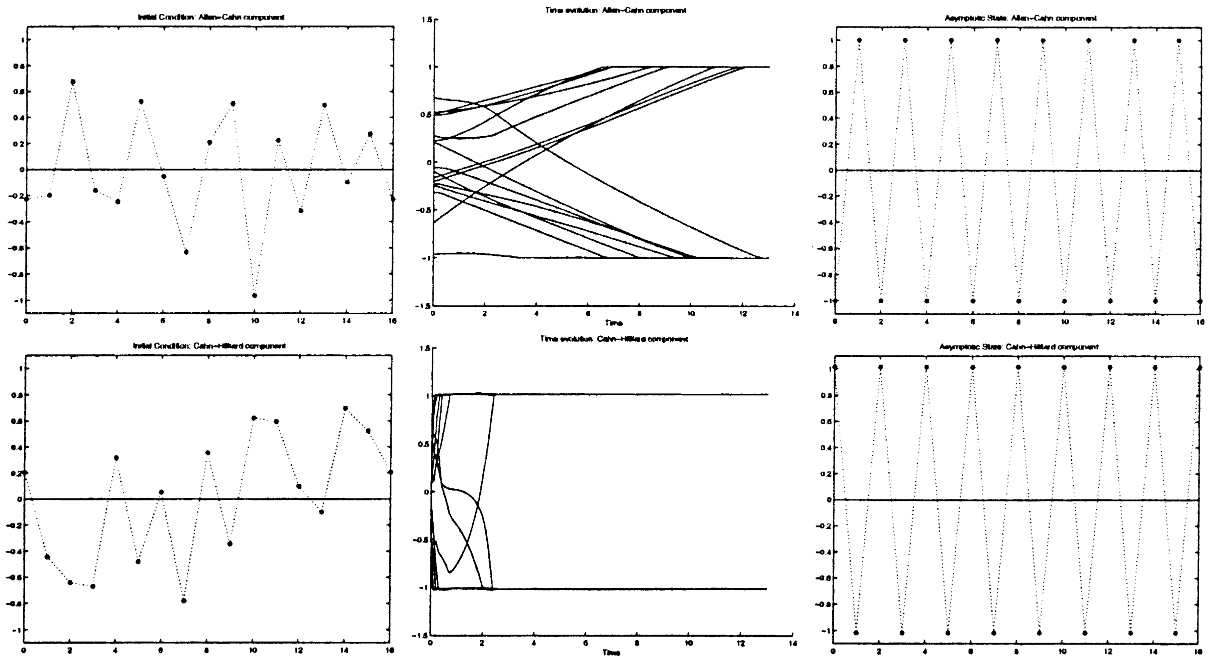

Fig. 7. Pattern formation in both components.

$$g_2^{\varepsilon}(u, v) = \begin{cases} \gamma_2 & \text{if } |v| < 1, \\ 0 & \text{if } |v| > 1. \end{cases}$$

In each of the following figures, the first two graphs give the initial conditions $u(0)$ and $v(0)$ on a periodic window. Note that $u_i(0)$, $v_i(0)$, $u_i(t)$ and $v_i(t)$ are only defined at the integers $i \in \mathbb{Z}$, but on the graphs we join adjacent components of u_i and v_i, respectively with a dotted line, as we find this makes the graphs easier to read. The second set of graphs in each figure give the time evolution of all the components $u_i(t)$ and $v_i(t)$ and the third set of graphs give the asymptotic states $\lim_{t\to} u_i(t)$ and $\lim_{t\to} v_i(t)$, where this is non-trivial.

In all cases we take random initial conditions such that the mass for the Allen–Cahn component $\sum_{i=1}^{N} u_i(0) = 0$ and the mass for the Cahn–Hilliard component $\sum_{i=1}^{N} v_i(0) = 0$, which is achieved by first taking truly random initial conditions then scaling and translating to achieve $M = 0$, while still ensuring that $|u_i| \leq 1$ and $|v_i| \leq 1$.

Note that although our concept of solution from Definition 2.1 is too weak to ensure mass conservation in the (Cahn–Hilliard) component in general, under tile periodic boundary conditions and point valued functions f_1^{ε} and f_2^{ε} considered in this section mass conservation is ensured.

Fig. 7 shows an example of pattern formation in both components, with $\beta_1 = 1$, $\gamma_1 = 4$, $\alpha = 1$, $\beta_2 = 6$, $\gamma_2 = -10$ and $s_1 = s_2 = 1$. In this case, the Cahn–Hilliard parameters are in the region marked v_C in Fig. 6 in which spatial chaos is possible, but since the Allen–Cahn parameters are in the region marked u_{B} in Fig. 5, the u component dominates the solution, giving a spatial entropy value of ln 1, and hence pattern formation.

Figs. 8 and 9 show an example of a parameter region in which pattern formation is obtained in the Allen–Cahn component, and spatially chaotic behavior in the Cahn–Hilliard component. Here, we take $\beta_1 = -4$, $\gamma_1 = -2$, $\alpha = 1$, $\beta_2 = -3$, $\gamma_2 = -10$ and $s_1 = s_2 = 1$ in both computations, and take different initial conditions. These parameter values correspond to the case mentioned in the previous section, where the spatial entropy is ln 1.555, and the only entries in the condensed transition matrix in this case are (i, j), with both $i, j < 16$, giving the constant

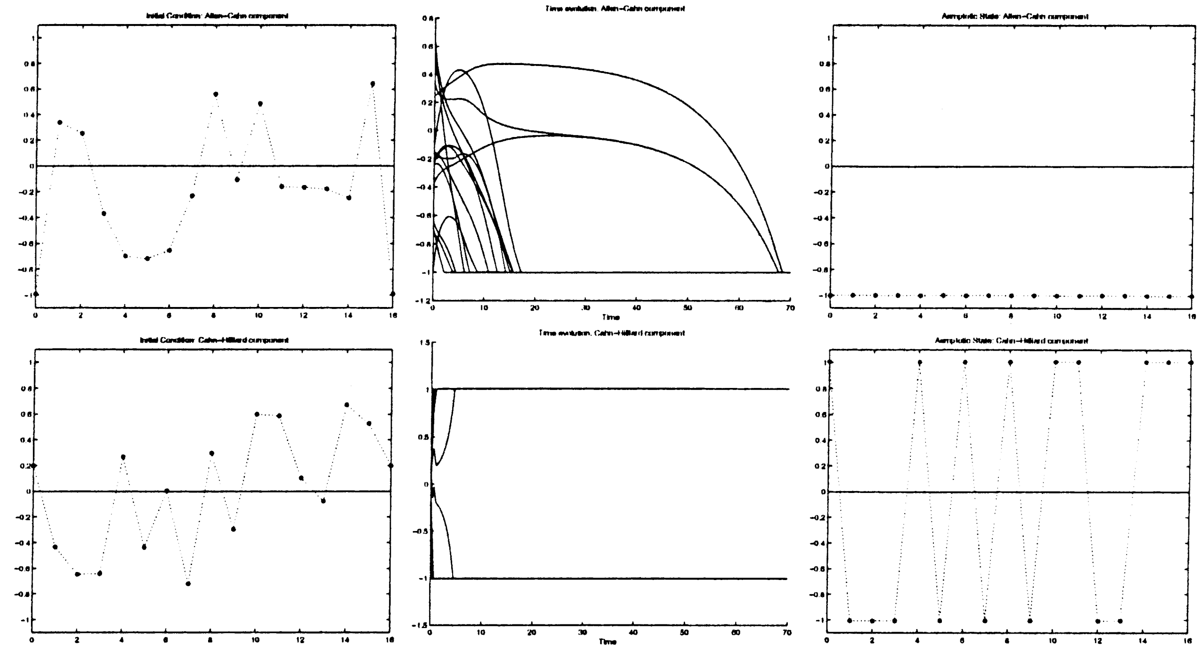

Fig. 8. Spatial chaos in v component only: example (i).

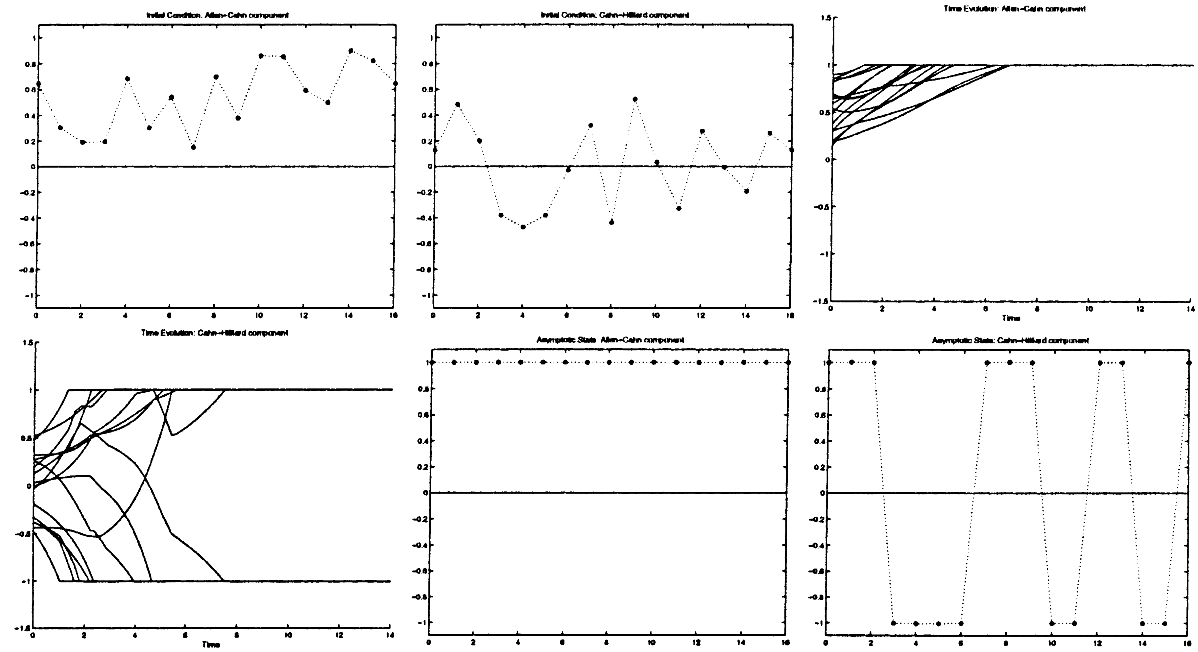

Fig. 9. Spatial chaos in Cahn–Hilliard component only: example (ii).

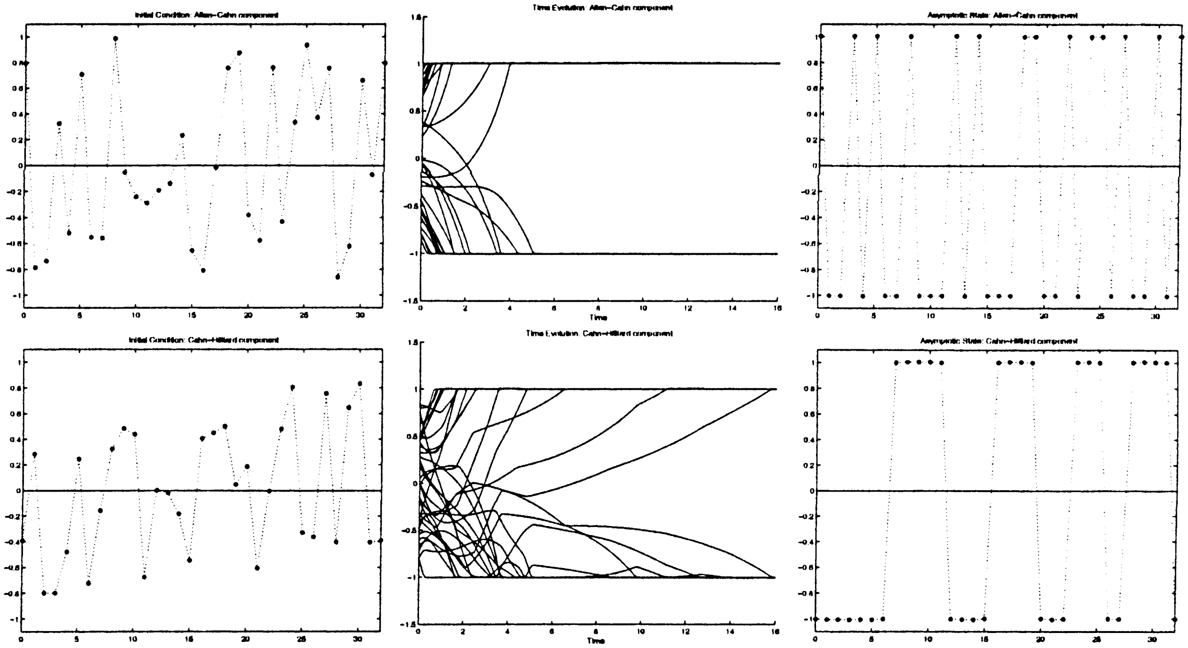

Fig. 10. Spatial chaos in both components: example (i).

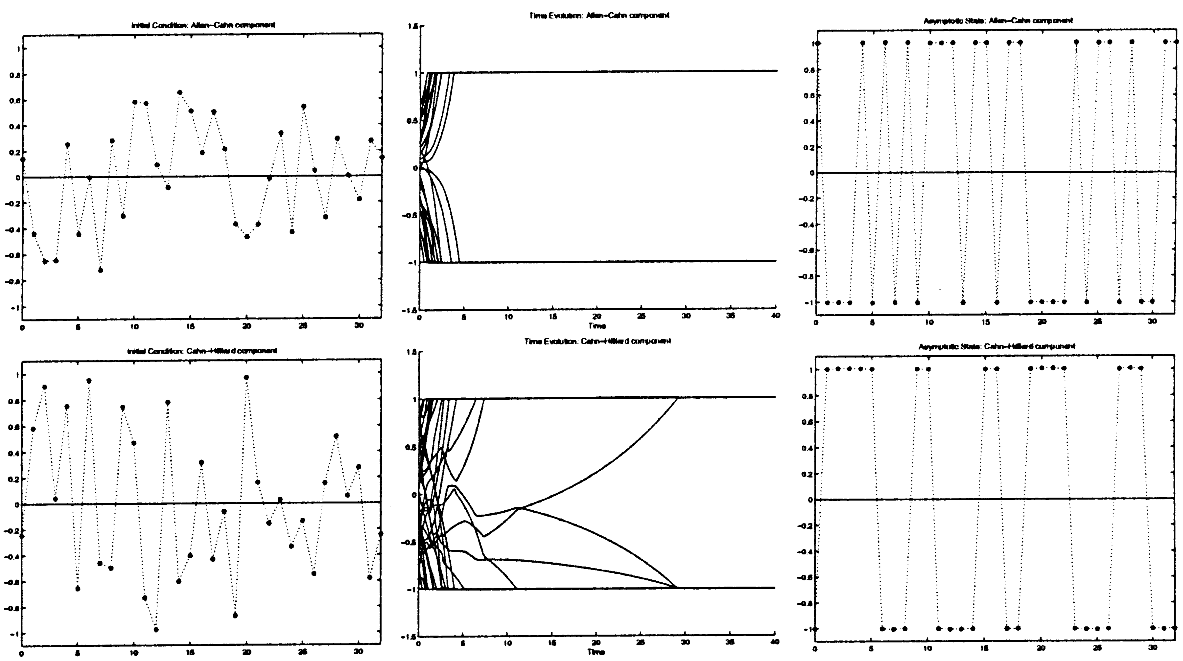

Fig. 11. Spatial chaos in both components: example (ii).

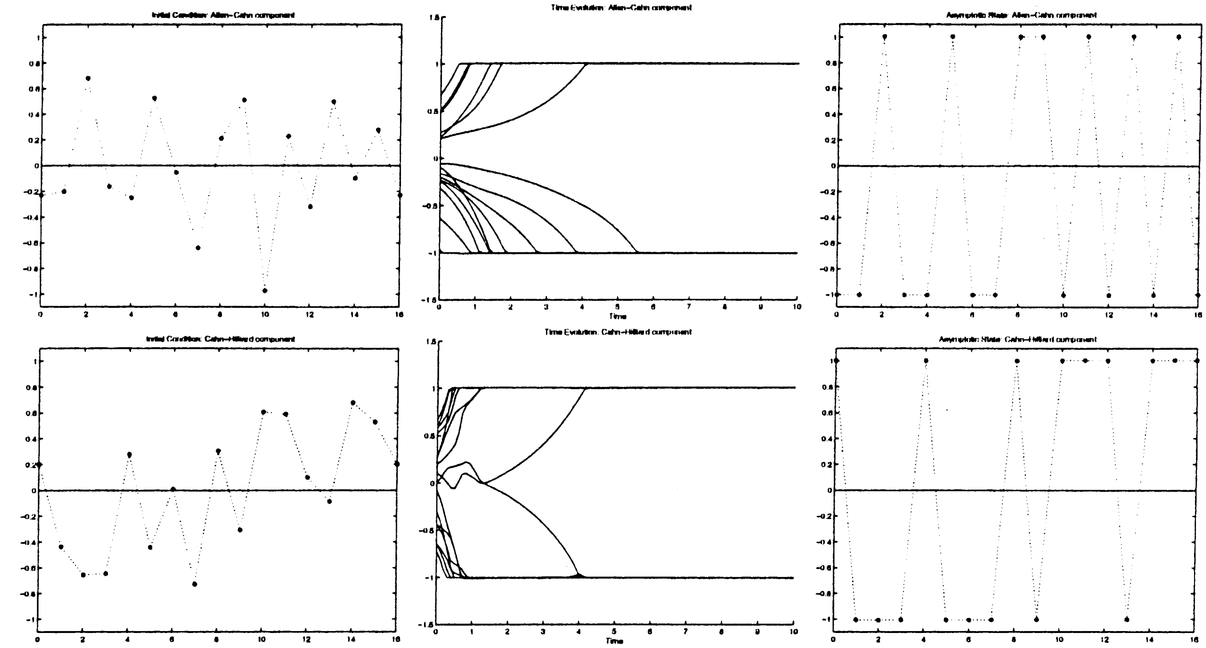

Fig. 12. Highest entropy value: example (i).

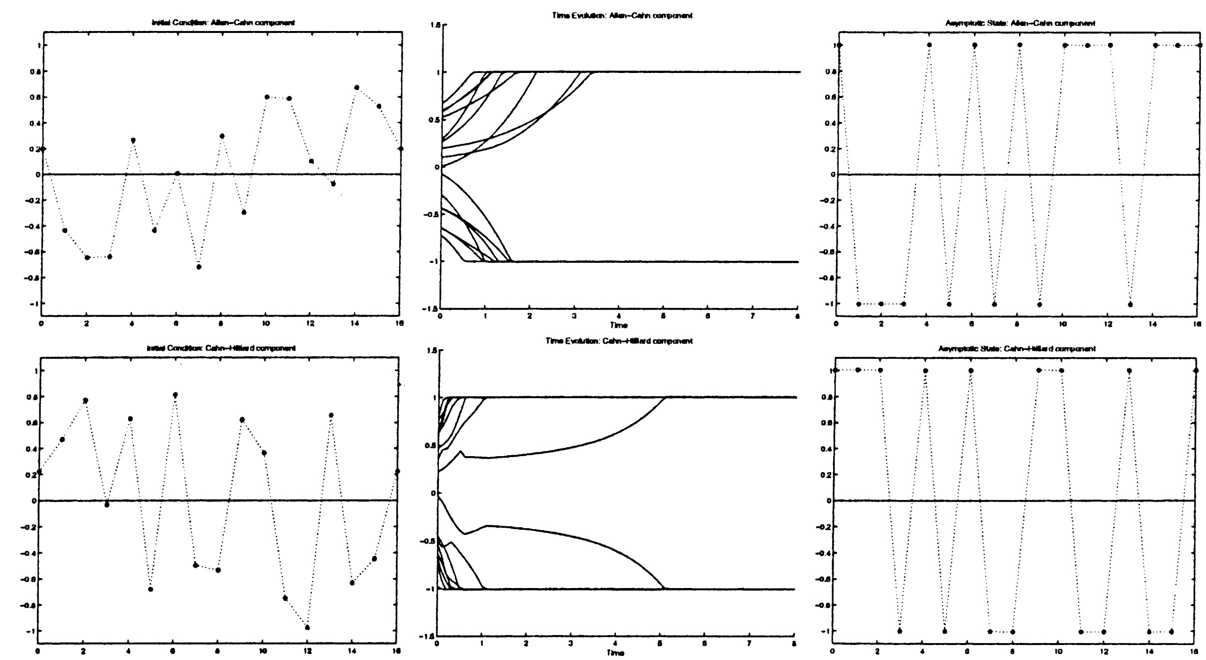

Fig. 13. Highest entropy: example (ii).

solution $u_i = 1$ for all $i \in \mathbb{Z}$ and (k, l) with $49 < k, l < 64$, giving the constant solution $u_i = -1$ for all $i \in \mathbb{Z}$. As suggested by these analytic results, in both computations the Allen–Cahn component converges to the constant solution, while the Cahn–Hilliard component converges to a different mosaic solution in each case.

Figs. 10 and 11 show computations on longer time intervals. In both computations we take $\beta_1 = 2$, $\gamma_1 = -2$, $\alpha = 1$, $\beta_2 = -4.9$, $\gamma_2 = -12.5$ and $s_1 = s_2 = 1$. In this case, the spatial entropy is ln 2, and the condensed transition matrix for this parameter region indicates that we should obtain spatially chaotic behavior in both components. In both computations, random initial conditions corresponding to $m = 0$ are taken, and the computations converge to different mosaic solutions in both components. Note that locally constant solutions appear in both components at some mesh points (for the Allen–Cahn component, a locally constant solution is such that $u_i = u_{i\pm1}$, as the solution at each point depends on the nearest-neighbors, while, since the Cahn–Hilliard component depends on the nearest and next-nearest-neighbors, a locally constant solution is $u_i = u_{i\pm1} = u_{i\pm2}$). However, for the reason already given, we do not obtain locally constant solutions at u_i and v_i for some $i \in \mathbb{Z}$ simultaneously.

In Figs. 12 and 13, we give a further example of spatially chaotic behavior in both components, by presenting computations in the parameter regions which give the highest entropy value in the coupled case. Here, $\beta_1 = 2$, $\gamma_1 = -2$, $\alpha = 1$, $\beta_2 = -2.5$, $\gamma_2 = -15$ and $s_1 = s_2 = 1$. In both computations, random initial conditions corresponding to $m = 0$ are taken, and the computations converge to different mosaic solutions.

7. Conclusion

We have considered a system of coupled spatially discrete Allen–Cahn and Cahn–Hilliard equations. Using a generalization of the double obstacle potential we have proven existence and uniqueness of solutions. Equilibrium solutions of mosaic type are considered and conditions are given for their existence and stability.

The spatial entropy is defined and explicit parameter dependent values are determined. An important outcome of this work is that the entropy provides a criteria for determining the extent to which the equations effectively act uncoupled, the extent to which a single equation in the system drives the system, and the extent to which the system is highly coupled but is not driven by a single equation. Figs. 5 and 6 may be used to determine the type of coupling for a given set of parameter values. Our measure of the type of coupling is based upon the spatial entropy and ultimately the existence and stability of mosaic solutions.

References

[1] K.A. Abell, Analysis and computation of the dynamics of spatially discrete phase transition equations, Ph.D. Thesis, University of Sussex (2001).
[2] K.A. Abell, A.R. Humphries, E.S. Van Vleck, Mosaic solutions and entropy for spatially discrete Cahn–Hilliard equations, IMA J. Appl. Math. 65 (2000) 219–255.
[3] S.M. Allen, J.W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Met. 27 (1979) 1085–1095.
[4] J.W. Barrett, J.F. Blowey, Finite element approximation of a degenerate Allen–Cahn/Cahn–Hilliard system, SIAM J. Numer. Anal., submitted for publication.
[5] J.F. Blowey, C.M. Elliott, The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy. Part I: Mathematical analysis, Eur. J. Appl. Math. 2 (1991) 233–280.
[6] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys. 28 (1958) 258–267.
[7] J.W. Cahn, J. Mallet-Paret, E.S. Van Vleck, Traveling wave solutions for systems of ODEs on a two-dimensional spatial lattice, SIAM J. Appl. Math. 59 (1999) 455–493.
[8] J.W. Cahn, A. Novick-Cohen, Evolution equations for phase separation and ordering in binary alloys, J. Statist. Phys. 76 (1994) 877–909.
[9] S.-N. Chow, J. Mallet-Paret, E.S. Van Vleck, Dynamics of lattice differential equations, Int. J. Bifurc. Chaos 6 (1996) 1605–1622.
[10] S.-N. Chow, J. Mallet-Paret, E.S. Van Vleck, Pattern formation and spatial chaos in spatially discrete evolution equations, Random Comput. Dyn. 4 (1996) 109–178.

[11] R. Dal Passo, L. Giacomelli, A. Novick-Cohen, Existence for an Allen–Cahn/Cahn–Hilliard system with degenerate mobility, Interf. Free Boundaries 1 (1999) 199–226.
[12] J.R. Dorgan, Spinodal decomposition in mixtures containing nematogens. II. Kinetics of spinodal decomposition, J. Chem. Phys. 98 (1993) 9094–9106.
[13] J.R. Dorgan, On the meaning of the spinodal in liquid crystalline solutions, Fluid Phase Equilibrium 109 (1995) 157–169.
[14] C.M. Elliott, A.R. Gardiner, I. Kostin, B. Lu, Mathematical and numerical analysis of a mean-field equation for the Ising model with glauber dynamics, in: P.E. Kloeden, K.J. Palmer (Eds.), Chaotic Numerics, Contemporary Mathematics, Vol. 172, American Mathematical Society, Providence, RI, 1994, pp. 217–242.
[15] P.C. Hohenberg, B.I. Halperin, Theory of dynamic critical phenomena, Rev. Mod. Phys. 49 (1977) 435–479.
[16] J.P. Laplante, T. Erneux, Propagation failure in arrays of coupled bistable chemical reactors, J. Phys. Chem. 96 (1992) 4931–4934.
[17] Y. Oono, S. Puri, Study of phase-separation dynamics by use of cell dynamical systems. I. Modeling, Phys. Rev. A 38 (1988) 434–453.
[18] A. Pazy, Semi-groups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin, 1983.
[19] C. Robinson, Dynamical Systems, CRC Press, Boca Raton, FL, 1995.
[20] A.M. Stuart, A.R. Humphries, Dynamical Systems and Numerical Analysis, Cambridge University Press, Cambridge, 1996.