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Weak finite-dimensional approximations of semi-linear elliptic PDEs with near-critical exponents

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Abstract. We make a formal study of the differential equation

$$u_{rr} + \frac{2}{r}u_r + \lambda u + u^{5+\varepsilon} = 0, \quad u_r(0) = u(1) = 0, \quad u > 0 \quad \text{if } 0 \leqslant r < 1, \tag{1}$$

when posed as a variational problem over a *finite-dimensional* subset S_h of H_0^1 comprising piecewise-linear functions defined on a mesh of size h. We determine critical points $U_h \in S_h$ of the variational form of (1). Such functions are perturbations of u when a solution of (1) exists, but we show that U_h can also exist when (1) has no solution and we determine an asymptotic expression for the solution branch (λ, U_h) when $\|U_h\|_\infty$ is large and $h\|U_h\|_\infty^2$ is small. If $\varepsilon = 0$, then u exists if $\lambda > \pi^2/4$, and we give a formula expressing U_h as a perturbation of u. If $\lambda \leqslant \pi^2/4$, then a solution of the differential equation does not exist, and U_h grows as $h \to 0$. We show that the rate of growth is proportional to $h^{-1/4}$ if $\lambda = \pi^2/4$, and $h^{-1/3}$ if $\lambda = 0$. We compare these results with estimates for the solutions of (1) when $\varepsilon \to 0^-$. Our results are obtained by using formal asymptotic methods – particulary the method of matched asymptotic expansions – and are supported by some numerical calculations.

1. Introduction

In a celebrated paper, Brezis and Nirenberg [7] opened the investigation of positive solutions of semi-linear elliptic partial differential equations of the form

$$\begin{cases} \Delta u + f(u;\lambda) = 0, \quad x \in \Omega \subset \mathbb{R}^n,\ n \geqslant 3 \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the parameterised function $f(u;\lambda)$ has the property that

$$f(u;\lambda)u^{-p_c} \to C \quad \text{as } u \to \infty, \tag{1.2}$$

where $C > 0$ is a constant and

$$p_{\mathrm{c}} \equiv \frac{n+2}{n-2}, \quad n \geqslant 3, \tag{1.3}$$

is called the critical Sobolev exponent for $\mathbb{R}^n$. Such a function is said to grow at a critical rate. The most subtle behaviour is observed in the case of three dimensions, so that $n = 3$ and

$$p_{\mathrm{c}} = 5, \tag{1.4}$$

and it is this case we will study in this paper. Until the study by Brezis and Nirenberg, the greater majority of the investigations of problem (1.1) were for functions growing at a sub-critical rate, so that $f(u;\lambda)u^{-p_{\mathrm{c}}} \to 0$ as $u \to \infty$. For this case, a fairly complete picture of the existence and uniqueness of the solutions of (1.1) has emerged for a wide variety of different domains Ω, and a survey of the results was presented in [22]. However, for functions growing at the critical rate (or, indeed, for functions growing at a super-critical rate such that $f(u;\lambda)u^{-p_{\mathrm{c}}} \to \infty$ as $u \to \infty$), things are very different. Here the existence, uniqueness and regularity of the solutions can change in a quite remarkable manner as the function $f(u;\lambda)$ varies with λ. For example, singularities in the solution may appear at critical values of λ. There have been many papers written since [7] (for example, [1,6,8,10,11,18,21,24,28,32]) which discuss these issues. Most of these papers restrict their discussions to symmetric solutions in the unit sphere and we extend these in this paper to the case of finite-dimensional solutions.

Questions about the solutions of (1.1) lie at the heart of many important problems in both pure and applied mathematics. Problems with critical exponent problems arise very naturally in studies of the curvature of manifolds and lead to deep links between the theories of differential geometry and partial differential equations [29]. Chandrasekhar [14] used such models to describe polytropic stars for which the nonlinearity is related to properties of the gas making up the star. Furthermore, general nonlinear functions arise in steady-state models of combustion (see the monograph by Bebernes and Eberly [5]), and frequently the function $f(u;\lambda)$ can grow very rapidly with u. Thus a theory (or, indeed, a numerical scheme) which restricts itself to sub-critically growing functions is far from complete.

Problem (1.1) can be cast in a variational form so that the solutions are critical points of the function $I(u)$ defined by

$$I(u) \equiv \int_{\Omega} \frac{1}{2}|\nabla u|^2 - F(u;\lambda)\,\mathrm{d}\Omega, \tag{1.5}$$

where

$$F(u) \equiv \int_0^u f(t;\lambda)\,\mathrm{d}t. \tag{1.6}$$

Here, the function $I(u)$ considered as a map $I : H_0^1(\Omega) \to \mathbb{R}$ is continuous if $F(u;\lambda)$ grows at a critical or sub-critical rate as condition (1.2) bounds $F(u)$ in terms of the $L^6(\Omega)$ norm of u which is, in turn, bounded by the $H_0^1(\Omega)$ norm. Furthermore, if $p < 5$ then the imbedding of the space $H_0^1(\Omega)$ into the space $L^{p+1}(\Omega)$ is *compact*. A direct consequence, when $f(u;\lambda)/u^p$ is bounded as u tends to infinity for some $p < 5$, is that a solution u of (1.1) may be found as a limit of a sequence of functions u_n tending toward a critical point of (1.5). Such a critical point is a zero of the functional $\Psi : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$\Psi(u)\varphi = -\int_\Omega \nabla u \cdot \nabla \varphi + f(u;\lambda)\varphi \, \mathrm{d}\Omega. \tag{1.7}$$

These compactness conditions are formalised in terms of the Palais–Smale compactness condition and the existence of solutions proven by the Mountain Pass Lemma; see, for example, the monograph by Chow and Hale [15].

If, in contrast, the function $f(u;\lambda)$ has critical or super-critical growth, then the Palais–Smale condition no longer holds and the standard variational existence proofs break down, although they can be extended in certain cases [7]. This is *not* simply a technicality. In an important paper, Pohozaev [25] derived an identity (stated in Section 2) satisfied by the solutions of (1.1). Applying this identity to the special case

$$f(u;\lambda) = u^p, \tag{1.8}$$

Pohozaev concluded that if Ω was a star-shaped domain (for example, the ball or any convex domain) then (1.1) has a solution for all $p < 5$ but *no* solution if $p \geqslant 5$.

If the nonlinear term is extended to

$$f(u;\lambda) = \lambda u^q + u^p, \quad 1 \leqslant q < 5,\ p \geqslant 5, \tag{1.9}$$

then the non-existence proof for (1.10), derived by Pohozaev, can be extended to show that there is a value of $\lambda_0 > 0$ such that no solutions of (1.1) exist if $\lambda < \lambda_0$, but that solutions may exist if $\lambda > \lambda_0$ [1,7].

Various attempts have been made to understand the above results in more detail by making perturbations to the differential equation problem (1.1), (1.9) such that a solution exists for the perturbed problem. This solution can then be studied as the perturbation tends to zero.

The perturbations include:

- Perturbing the nonlinear term to
$$f(u;\lambda) = \lambda u^q + u^p, \quad p < 5,\ p \approx 5 \tag{1.10}$$
and studying the limit of the solutions as $p \to 5$.
- Perturbing the lower-order term so that $\lambda = \lambda_0 + \delta$ with δ small.
- Perturbing the domain so that the perturbed domain is not star-shaped. For example, introducing a small hole and letting the hole diameter tend towards zero.

For the general problem (1.1), (1.9), we observe that as the perturbation tends to zero then the perturbed solution converges to the true solution, when such exists, and if not (for example, if $\lambda = 0$ and $p \to 5^-$) then the solution of the perturbed problem forms a singularity in the limit.

In this paper, we consider a new perturbation to problem (1.1), (1.9). To do this we pose (1.5) as a function over a finite-dimensional space $S_h \subset H_0^1(\Omega)$ and find the critical points in S_h of this function. Equivalently, we find a zero $U_h \in S_h$ of the functional $\Psi(U_h)$ defined in (1.7) and now considered as a map from S_h to its dual space, and then study U_h as the dimension $(=1/h)$ of S_h increases.

This perturbation is interesting in its own right and is also the basis of the finite-element method for finding numerical solutions of (1.1) [16].

As S_h is finite-dimensional, the imbedding of S_h into $L^{p+1}(\Omega)$ is compact for all $p > 0$. Consequently, we can show that there is a wide class of nonlinear functions $f(u;\lambda)$ (including (1.9)) for which (1.5) has a critical point in S_h. Significantly, Pohozaev's result does not exclude the existence of such functions. We may thus determine a set of such critical points in a sequence of finite-dimensional spaces of increasing dimension and look at the limit of these solutions. Again, if the underlying problem has a solution then we see convergence to this solution, and if not then the finite-dimensional solutions become unbounded in the limit.

The existence of such finite-dimensional solutions when (1.1) does not have a solution is of interest in the numerical analysis of (1.1), as such solutions may be confused as being approximations of the true solution and give a misleading picture of the interval of existence. We call such finite-dimensional solutions *spurious*. Their detection and elimination (for example, by changing the computational mesh) is an important part of any numerical investigation of (1.1), and we discuss the numerical consequences of our results in more detail in the related paper [12].

The purpose of this paper is to study a branch (λ, U_h) of these finite-dimensional solutions when Ω is the unit sphere and to obtain estimates on the behaviour of U_h (including the growth of the spurious solutions) in terms of the dimension of S_h. In Section 2 we show that there are close and interesting symmetries between these estimates and the corresponding ones for solutions of the other perturbed problems described above.

A difficulty of our approach is that the tight analytic estimates on the solution of (1.1) which follow from Pohozaev's result cannot immediately be determined for the finite-dimensional solutions which do not satisfy this identity. Our approach to the study of U_h consequently differs from the rigorous methods (presented, for example, in [8]), and we use formal methods based upon the method of matched asymptotic expansions to obtain our estimates. By performing some numerical estimates we demonstrate that our results are sharp.

Thus, suppose we consider the radially symmetric solutions of (1.1), (1.9) in a unit sphere in $\mathbb{R}^3$, with $f(u;\lambda)$ defined by

$$f(u;\lambda) = \lambda u + u^{5+\varepsilon}, \tag{1.11}$$

so that

$$u(\mathbf{x}) = u\big(|\mathbf{x}|\big) \equiv u(r) \quad \text{with } r = |\mathbf{x}|.$$

Such a solution then satisfies the ordinary differential equation

$$\begin{cases} u_{rr} + (2/r)u_r + \lambda u + u^{5+\varepsilon} = 0, \\ u(r) > 0 \quad \text{if } 0 \leqslant r < 1, \quad u_r(0) = u(1) = 0. \end{cases} \tag{1.12}$$

It is shown in [7] that if $\varepsilon \geqslant 0$ then (1.12) has a non-trivial solution branch (of monotone decreasing solutions $u(r)$) which exists only if $\lambda \in (\lambda_0, \pi^2)$, where $\lambda_0 \geqslant \pi^2/4$. Furthermore, if $\varepsilon = 0$ then $\lambda_0 = \pi^2/4$, $\|u\|_\infty = u(0) \to \infty$ as $\lambda \to \pi^2/4$, and the function $u(r)$ forms a singularity at $r = 0$. In contrast, if $\varepsilon < 0$ then solutions $u_\varepsilon(r)$ exist for all $\lambda < \pi^2$, but if $\lambda \leqslant \pi^2/4$ these become singular at $r = 0$ as $\varepsilon \to 0^-$. We show that if $\varepsilon = 0$ and U_h is the positive solution of an appropriate finite-dimensional perturbation of (1.12) then similarly U_h exists for all $\lambda < \pi$, and if $\lambda < \pi^2/4$ then $\|U_h\|_\infty$ becomes singular as the dimension ($=1/h$) of the approximating space increases.

To construct such a finite-dimensional perturbation we consider the weak formulation of (1.12). To do this we introduce an inner product defined by

$$\langle u, v\rangle \equiv \int_0^1 u(r)v(r)r^2 \,\mathrm{d}r, \tag{1.13}$$

and consider the associated Sobolev space H_0^1, comprising those functions which vanish at $r = 1$ with (finite) norm defined

$$\|u\|^2_{H_0^1} = \langle u', u'\rangle.$$

A weak solution of (1.12) is then any function $u(r) \in H_0^1$ which satisfies

$$\Psi(u)\varphi \equiv -\langle u', \varphi'\rangle + \lambda\langle u, \varphi\rangle + \left\langle u^{5+\varepsilon}, \varphi\right\rangle = 0, \quad \forall\varphi \in H_0^1. \tag{1.14}$$

Here primes denote differentiation with respect to r.

We define a finite-dimensional subspace $S_h \subset H_0^1$ of dimension $N \equiv 1/h$ to be the space spanned by continuous *piecewise-linear* functions, constructed on a uniform mesh of size h (these functions are defined in Section 3). Now, let $U_h(r)$ be a non-trivial member of S_h satisfying

$$-\left\langle U_h', \varphi_h'\right\rangle + \lambda\langle U_h, \varphi_h\rangle + \left\langle U_h^{5+\varepsilon}, \varphi_h\right\rangle = 0, \quad \forall\varphi_h \in S_h. \tag{1.15}$$

The function U_h is generally called the *finite-element* approximation to u. In Section 3 we show that, if the underlying solution u to (1.12) exists and is isolated, then

$$\|U_h - u\|_{H_0^1} \to 0, \quad \|U_h - u\|_\infty \to 0 \quad \text{as } h \to 0.$$

In [23], global bifurcation theory [27] is applied to show that problem (1.15) has a continuous solution branch (λ, U_h), parametrised by λ which bifurcates from the trivial solution at $\lambda = \lambda_{1,h} > \pi^2$. Positive solutions exist on this branch iff $\lambda < \lambda_{1,h}$, and these satisfy the bound

$$\|U_h\|_\infty < K(\varepsilon)h^{2/(4+\varepsilon)}, \tag{1.16}$$

where K depends only upon ε. (There may also be other solutions of (1.15) which do not lie on this branch, however, there is no numerical evidence so far for these, and as they are quite unrelated to the underlying solution we do not consider them in this paper.)

In Fig. 1 we present a sequence of such solution branches for decreasing values of h when $\varepsilon = 0$. In this figure, several important features of the solution branch are visible. In particular, the finite-dimensional solution exists for all h and is unique even when $\lambda < \pi^2/4 = 2.4674011$. Furthermore, if λ is fixed then $\|U_h\|_\infty$ increases monotonely as $h \to 0$, or if h is fixed, as $\lambda \to 0$. Finally, the solution is large (and, consequently, $f(u;\lambda)$ is very large) for a range of values of λ including values greater than $\pi^2/4$. It is significant that none of the numerical computations shows a singularity developing at $\lambda = \pi^2/4$ or, indeed, looks anything other than regular at this point.

Our aim in this paper is to obtain a formal asymptotic explanation for these results and to determine the behaviour of those solutions for which h is small and U_h is large. In particular, if we define

$$\gamma \equiv \|U_h\|_\infty = U_h(0) \quad \text{and} \quad H \equiv h\gamma^2, \tag{1.17}$$

our results apply to those solutions for which

$$\gamma \gg 1 \quad \text{and} \quad H \ll 1. \tag{1.18}$$

We now state our formal asymptotic results.

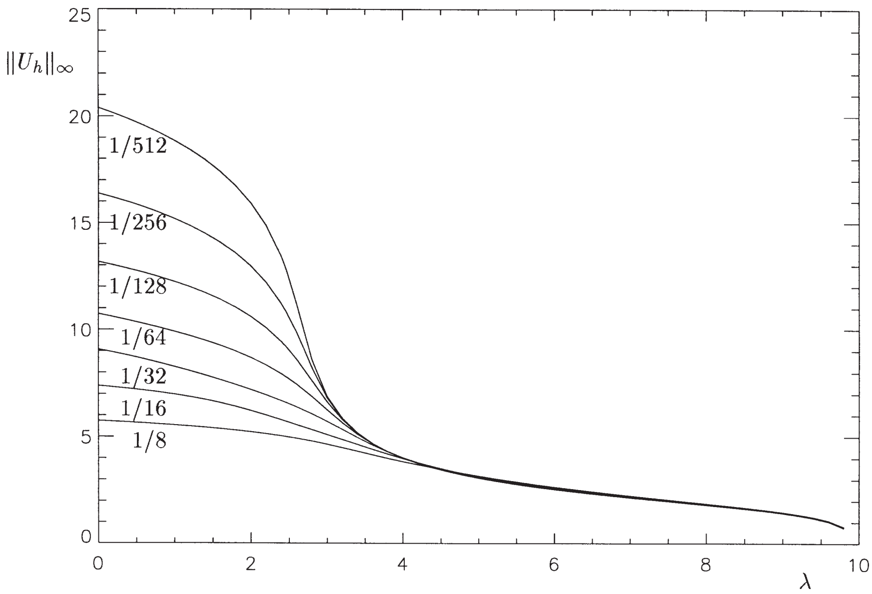

Fig. 1. Bifurcation diagrams for different meshes with h varying from $1/8$ to $1/512$.

Formal Proposition 1.1. *If conditions* (1.17), (1.18) *are satisfied and* $|\varepsilon \log^2(\gamma)|$ *is small, then*

$$\left[Ah^2\gamma^5 - \lambda\sqrt{3}\pi\gamma^{-3} - \frac{\pi\varepsilon\gamma}{32\sqrt{3}}\right]\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\frac{1}{\gamma^2}\right) + \mathrm{O}\left(\varepsilon \log^2(\gamma)\right)\right)$$
$$= \frac{1}{\gamma}\theta(\lambda)\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\frac{1}{\gamma^4}\right)\right), \tag{1.19}$$

where

$$\theta(\lambda) = \sqrt{|\lambda|}\cot\left(\sqrt{|\lambda|}\right) \quad \textit{if } \lambda \neq 0, \qquad \theta(0) = 1, \tag{1.20}$$

and

$$A = \frac{13}{\sqrt{3}}\frac{\pi}{4608}\left(1 + \mathrm{O}(H)\right). \tag{1.21}$$

Formula (1.19) provides a unified asymptotic description of the finite-dimensional solution branch both for values of λ for which U_h converges to u as $h \to 0$ and for values at which it is spurious. In particular, the character of the solutions changes completely when $\theta(\lambda)$ changes sign at $\lambda = \pi^2/4$. As a direct consequrnce of the above proposition, we have

Formal Proposition 1.2. *If* $\varepsilon = 0$ *and* (1.17), (1.18) *are satisfied, then*

(i) *if* λ *is larger than and close to* $\pi^2/4$, *then*

$$\|u\|_\infty = \frac{\gamma}{(1 - Ch^2\gamma^8)^{1/2}}\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(1/\gamma^2\right)\right), \tag{1.22}$$

where u *solves* (1.12) *and*

$$C = \frac{13}{3456\pi^2}\left(1 + \mathrm{O}(H)\right);$$

(ii) *if* $\lambda = \pi^2/4$, *then as* $h \to 0$,

$$\gamma = h^{-1/4} C^{-1/8}\big(1 + \mathrm{O}\big(h^{1/2}\big)\big), \tag{1.23}$$

and the function U_h *forms a peak at the origin with width proportional to* $h^{1/2}$;

(iii) *if* $\lambda < \pi^2/4$, *then as* $h \to 0$,

$$\gamma = h^{-1/3}\left(\frac{\theta(\lambda)}{A}\right)^{1/6}\big(1 + \mathrm{O}\big(h^{2/3}\big)\big), \tag{1.24}$$

and the function U_h *forms a peak at the origin with width proportional to* $h^{2/3}$.

We note, that for small h, the singular growth of U_h in (1.23), (1.24) is consistent with bounds (1.18) for both H and γ.

The estimates above are strongly supported by some numerical experiments, reported in Section 5, and are much sharper than the bound on U_h given in (1.16).

Our formal results follow from a close study of the singularity in $u(r)$ which arises when $\varepsilon = 0$ and $\lambda \to \pi^2/4$. Suppose now that $u(0) \equiv \|u\|_\infty = \gamma$. Then, if γ is large and $|\varepsilon|$ is small, $\lambda u + u^{5+\varepsilon} \approx u^5$. In this case, (1.12) is approximated by the ordinary differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}w}{\mathrm{d}r} + w^5 = 0, \quad w_r(0) = 0, \quad w(0) = \gamma. \tag{1.25}$$

For all γ, this equation has the strictly positive monotone decreasing solution $w_\gamma(r)$ defined by

$$w_\gamma(r) \equiv \frac{\gamma}{(1 + \gamma^4 r^2/3)^{1/2}} \tag{1.26}$$

such that $w_\gamma(0) = \gamma$. Close to the origin (where both u and w_γ are large), u is closely approximated by w_γ [21]. (The function w_γ plays an important role in the theory of instantons and in obtaining imbedding estimates in Sobolev spaces [7].)

In the finite-dimensional problem, we compare $U_h(r)$ (with $U_h(0) = \gamma$), with the piecewise-linear function $W_{h,\gamma}(r)$ defined by $W_{h,\gamma}(ih) = w_\gamma(ih)$ for each integer i such that $ih \in [0,1]$. Standard results in the theory of the interpolation of functions [16] state that the relative L_∞ error e between w_γ and $W_{h,\gamma}$ is bounded by

$$e < Kh^2 \big\|(w_\gamma)_{rr}\big\|_\infty / \|w_\gamma\|_\infty,$$

where K is a constant independent of w_γ. Using the functional form of w_γ we may estimate these terms to give

$$e = \mathrm{O}\big(h^2\gamma^4\big) = \mathrm{O}\big(H^2\big).$$

Thus, provided that H is sufficiently small then $W_{h,\gamma}$ is a close approximation to w_γ. The formal estimates then follow from using the method of matched asymptotic expansions to describe U_h as a perturbation of $W_{h,\gamma}$. A somewhat similar approach, using matched asymptotic expansions to derive numerical error estimates for problems with singularities, has been considered by Ward and Keller [31].

We may deduce many interesting further results from (1.19) when ε is non-zero, noting that we have existence of solutions for all λ when $\varepsilon < 0$.

Formal Corollary 1.3. *Suppose that* $\varepsilon < 0$ *is small in modulus and* U_h *satisfies* (1.15); *then as* $h \to 0$,

(i) *if* $\lambda = 0$, *then* γ *satisfies*

$$-\varepsilon\gamma^2 = \frac{32\sqrt{3}}{\pi}\left(1 - Ah^2\gamma^6\right)\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(1/\gamma^2\right) + \mathrm{O}\left(\varepsilon \log^2(\gamma)\right)\right); \tag{1.27}$$

(ii) *if* $\lambda = \pi^2/4$, *then*

$$-\varepsilon\gamma^4 = 24\pi^2\left(1 - Ch^2\gamma^8\right)\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(1/\gamma^2\right) + \mathrm{O}\left(\varepsilon \log^2(\gamma)\right)\right). \tag{1.28}$$

We compare these estimates with the results of Atkinson and Peletier [2] (see also [8]) that, in the respective cases $\lambda = 0$ and $\lambda = \pi^2/4$,

$$-\varepsilon\|u\|_\infty^2 \to \frac{32\sqrt{3}}{\pi} \quad \text{and} \quad -\varepsilon\|u\|_\infty^4 \to 24\pi^2 \quad \text{as } \varepsilon \to 0. \tag{1.29}$$

Quite different behaviour is observed if $\varepsilon > 0$.

Formal Corollary 1.4. *If* $0 < \varepsilon \ll 1$, *then for all* λ *such that* γ *is large,*

$$\gamma = h^{-1/2}\left(\frac{\pi\varepsilon}{32A\sqrt{3}}\right)^{1/4}\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\frac{1}{\gamma^2}\right) + \mathrm{O}\left(\varepsilon \log^2(\gamma)\right)\right) \quad \text{as } h \to 0. \tag{1.30}$$

We observe that estimate (1.30) is consistent with (1.18) provided that ε is small. Furthermore, (1.30) is directly comparable with the bound given by (1.16) if

$$K(\varepsilon) = \left(\frac{\pi\varepsilon}{32A\sqrt{3}}\right)^{1/4}.$$

If $\varepsilon = 0$ then estimate (1.30) for γ is degenerate, and (1.24) holds with the growth rate in h changing from $h^{-1/2}$ to $h^{-1/3}$ if $\lambda = 0$, or to $h^{-1/4}$ if $\lambda = \pi^2/4$. This mirrors the discontinuous change in the growth rate of the solutions as functions of ε given in (1.29).

The layout of the remainder of this paper is as follows. In Section 2 we give a brief outline of the existing theory for the solutions of (1.1), (1.11) in symmetric domains and compare the results obtained with those for the finite-dimensional perturbation. In Section 3 we derive a finite-element method for the solution of (1.12) and describe some existing results on the convergence and *a priori* error estimation of the solutions. In Section 4 we derive the formal results quoted in Propositions 1.1 and 1.2 and the corollaries. In Section 5 we make some numerical computations which show remarkably strong support for the formal results obtained in Section 4. Furthermore, we look at the behaviour of the Pohozaev functional for the finite-dimensional solutions. Finally, in Section 6 we briefly discuss how these results can be extended to more general domains and draw some conclusions from this work. More details of the latter calculation will be given in the forthcoming paper [12].

2. Results for the continuous problem

We now present some of the known results for the smooth solutions of problems (1.1), (1.12) and their perturbations which we can compare with the behaviour of the finite-dimensional solutions of (1.15).

Firstly, we state Pohozaev's result. Suppose that the general function u, defined in $\Omega \subset R^3$, satisfies the conditions that $uf(u;\lambda)$, $F(u;\lambda)$ are integrable over Ω, and that $|\nabla u|$ is defined and square-integrable on $\partial\Omega$. (Here $F(u;\lambda) = \int_0^u f(v;\lambda)\,\mathrm{d}v$). We define the Pohozaev functional $P(u;\lambda)$ by

$$P(u;\lambda) = \int_\Omega \frac{1}{2}uf(u;\lambda) - 3F(u;\lambda)\,\mathrm{d}\Omega + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (r\cdot\nu)\,\mathrm{d}S, \tag{2.1}$$

where r is the position vector and ν is the outward pointing normal to the surface $\partial\Omega$. Thus, if Ω is the unit sphere and $u \equiv u(r)$ is radially symmetric, then

$$P(u;\lambda) = 4\pi\left(\int_0^1 \left[\frac{1}{2}uf(u;\lambda) - 3F(u;\lambda)\right] r^2\,\mathrm{d}r + \frac{1}{2}u_r^2\right), \tag{2.2}$$

so that if $f(u) = \lambda u + u^5$ then

$$P(u;\lambda) = 4\pi\left(-\int_0^1 \lambda u^2 r^2\,\mathrm{d}r + \frac{1}{2}u_r^2\right). \tag{2.3}$$

It is then shown in [25] that if u is a solution of (1.1) then

$$P(u;\lambda) = 0, \tag{2.4}$$

and this result is used to prove most of the results in this section. However, we show in Section 5 that identity (2.4) does not appear to hold for the finite-dimensional weak solution U_h of (1.12).

We now briefly state results on perturbations of both the partial differential equation (1.1) and of the ordinary differential equation (1.12).

2.1. Perturbations of the nonlinear term

The problem of studying the limit of (1.1), (1.10) in the limit of $p \to 5$ was raised in [2]. Asymptotic results were presented by Budd [10] and subsequently proved in [8]. These results were for symmetric solutions in spheres, and some conjectures were raised for solutions in more general domains. Some of these conjectures have subsequently been answered in the papers of Rey [28] and Han [21] (see also [32]). We firstly consider the case of $\lambda = 0$, for which the following result is known.

Theorem 2.1. *Let u_ε be a solution of*

$$\Delta u_\varepsilon + u_\varepsilon^{5+\varepsilon} = 0, \quad \varepsilon < 0, \tag{2.5}$$

with $u_\varepsilon = 0$ on the boundary of Ω a smooth, bounded (*star-shaped*) *domain in R^3; then*

(i) *there is a constant K* (*related to the Green's function of the domain*) *such that*

$$\lim_{\varepsilon\to 0} -\varepsilon\|u_\varepsilon\|_\infty^2 = K; \tag{2.6}$$

(ii) *there is a critical point $\mathbf{x}_0$ such that*

$$\lim_{\varepsilon\to 0} u_\varepsilon(\mathbf{x}_0) = \infty, \qquad \lim_{\varepsilon\to 0} u_\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \neq \mathbf{x}_0.$$

For the unit sphere, $\mathbf{x}_0$ is the origin and

$$K = \frac{32\sqrt{3}}{\pi}. \tag{2.7}$$

The above results are proved by showing that the function $u(\mathbf{x})$ away from $\mathbf{x}_0$ is a perturbation of the Green's function of the domain, and close to $\mathbf{x}_0$ it is a perturbation of the function $w_\gamma(r)$ (with $r = |\mathbf{x} - \mathbf{x}_0|$), given in (1.26). Indeed, the convergence of $u(\mathbf{x})$ to $w_\gamma(r)$ as $\gamma \to \infty$ over compact subsets in the variable $\mathbf{y} = (\mathbf{x} - \mathbf{x}_0)/\gamma^2$ is proven in [21] for quite general domains. Estimates for $\|u\|_\infty$ then follow from substituting these two representations for u into the Pohozaev identity (2.4).

For the sphere, we also have the inequality

$$u(r) \leqslant w_\gamma(r),$$

which can be used to obtain further estimates.

For large γ, the function $w_\gamma(r)$ resembles the delta function, having a narrow peak of 'width' γ^{-2} near $r = 0$ and vanishing as $\gamma \to \infty$ for any $r \neq 0$. A numerical approximation of this behaviour forms the basis of our asymptotic calculation in Section 4.

Although u tends to infinity in the L_∞ norm, it is bounded in the H_0^1 norm and even tends to zero in the L_2 norm as $\varepsilon \to 0$. Indeed, for quite general domains,

$$\|u\|_{H_0^1} \to \frac{3}{8}\sqrt{3}\pi^2 \quad \text{and} \quad \|u\|_{L_2} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.8}$$

In the unit sphere, the following precise estimates have been obtained by Brezis and Peletier [8] for the more general problem:

$$u_{rr} + \frac{2}{r}u_r + \lambda u + u^{5+\varepsilon} = 0, \quad u_r(0) = u(1) = 0. \tag{2.9}$$

Theorem 2.2. *Let u_ε satisfy* (2.9) *and let $\lambda < \pi^2/4$; then, as $\varepsilon \to 0$, a singularity forms at $r = 0$ such that*

$$\lim_{\varepsilon \to 0} -\varepsilon u^2(0) = \frac{32\sqrt{3}}{\pi}\theta(\lambda), \tag{2.10}$$

where $\theta(\lambda)$ is defined in (1.20)*. If $r \neq 0$ then*

$$u(r) \to \frac{\sqrt{3}}{u(0)} 4\pi G_\lambda(r),$$

where

$$G_\lambda(r) = \frac{1}{4\pi r} + g(r),$$

with $g(r)$ a regular function, is the Green's function for the operator $-\Delta - \lambda$.

When $\lambda = \pi^2/4$, estimate (2.10) no longer applies as $\theta(\pi^2/4) = 0$, and instead we have

$$\lim_{\varepsilon \to 0} -\varepsilon u_\varepsilon^4(0) = 24\pi^2, \tag{2.11}$$

and, if $r \neq 0$, then

$$u(r) \to \frac{\sqrt{3}}{u(0)} \frac{\cos(\pi r/2)}{r}. \tag{2.12}$$

When $\lambda > \pi^2/4$, the function $u(r)$ tends to a non-singular limit as $\varepsilon \to 0$.

Thus the growth rate of the solutions changes discontinuously from being of $O(\varepsilon^{-1/2})$ if $\lambda < \pi^2/4$ to $O(\varepsilon^{-1/4})$ at $\lambda = \pi^2/4$.

This change mirrors the corresponding change in the growth rate of the finite-dimensional solutions from $O(h^{-1/4})$ to $O(h^{-1/3})$ at precisely the same value of λ.

Informally, formula (1.19) gives a unified description of both of these types of behaviour, visible by respectively substituting $h = 0$ and calculating γ in the limit $\varepsilon \to 0$, or substituting $\varepsilon = 0$ and calculating γ in the limit of $h \to 0$.

2.2. Perturbations to λ

If $\lambda = \pi^2/4+\delta,\ \varepsilon = 0$, then the the solution of (2.9) exists if $\delta > 0$ and becomes singular as $\delta \to 0$. The nature of this singularity was investigated formally by Budd [10] and rigorously by Brezis and Peletier [8]. In particular, we have

Theorem 2.3. *If* $\varepsilon = 0$ *and* $\lambda = \pi^2/4 + \delta,\ \delta > 0$, *then as* $\delta \to 0$ *a singularity forms at the origin and*

$$\lim_{\delta\to 0} \delta u^2(0) = \frac{\sqrt{3}\pi^3}{2}. \tag{2.13}$$

Again, $u(r)$ is closely approximated by, and bounded above by the function $w_\gamma(r)$. Result (2.13) can also be formally extended [10] to the case $\varepsilon > 0$, where it was proved in [11] that (1.12) can have multiple solutions for certain values of λ. Formula (1.19) again allows us to unify this result with those of the finite-dimensional and small ε perturbations by determining γ for those values of λ for which $|\theta(\lambda)|$ is small.

2.3. Perturbations to the domain

Finally, we consider perturbations to problem (2.9) where we pose (1.1) on the annulus and take boundary conditions

$$u(a) = u(1) = 0, \quad a > 0,\ a \ll 1. \tag{2.14}$$

As $a \to 0$ then similar behaviour in u to that of the previous sections is observed. In particular, it is shown in [3,4,13] that the following theorem is true:

Theorem 2.4. *Let* $\lambda = 0$; *then the solution of problem* (2.9) *with the boundary conditions* (2.14) *satisfies*:

(i) *if* $\varepsilon = 0$, *then there exists a constant* $\gamma(\varepsilon)$ *such that* $\gamma(\varepsilon) \to \infty$ *as* $\varepsilon \to 0^-$ *for which*

$$\|u\|_\infty \to \gamma(\varepsilon) < \infty \quad \textit{as } a \to 0;$$

(ii) *if* $\varepsilon = 0$, *then*

$$a^{1/4}\|u\|_\infty \to 3^{1/4} \quad \text{as } a \to 0;$$

(iii) *if* $\varepsilon > 0$, *then there exists a constant* $A(\varepsilon)$ *such that* $A(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$ *for which*

$$a^{2/(4+\varepsilon)}\|u\|_\infty \to A(\varepsilon) \quad \text{as } a \to 0.$$

There are again striking similarities between this and the other perturbations, in particular the discontinuous change in the growth of the solutions from $a^{-2/(4+\varepsilon)}$ when $\varepsilon > 0$ to $a^{1/4}$ when $\varepsilon = 0$. We note, also, that in each of the critical cases we observe a growth rate of $1/4$ in the solution as the relevent perturbation tends to zero.

3. A finite-element method and its convergence

In this section we give some preliminary results on the existence of solutions U_h of (1.15) and of the convergence of U_h to u when u exists.

For the purposes of this paper, we consider those discretisations of (1.12) over the space S_h spanned by the piecewise-linear functions $\varphi_i(r)$ for $i = 0, \ldots, N-1$, $N = 1/h$ defined by

$$\varphi_0(r) = 1 - \frac{r}{h}, \quad r \in [0, h], \qquad \varphi_0(r) = 0, \quad \text{otherwise}, \tag{3.1}$$

and

$$\varphi_i(r) = \bigl(r - (i-1)h\bigr)/h, \quad r \in [(i-1)h, ih],$$
$$\varphi_i(r) = \bigl((i+1)h - r\bigr)/h, \quad r \in [ih, (i+1)h],$$

with $\varphi_i(r) = 0$, otherwise. As these then span S_h, problem (1.15) has a solution U_h iff

$$-\langle U_h', \varphi_i' \rangle + \langle f(U_h), \varphi_i \rangle = 0, \quad i = 0, \ldots, N-1. \tag{3.2}$$

As a first result, we give a proof of the existence of solutions of (3.2) for a range of functions $f(u; \lambda)$.

Theorem 3.1. *Let* $f(u; \lambda)$ *satisfy*

$$f(u; \lambda) = \lambda u^q + u^p, \quad p > q \geqslant 1;$$

then, if $q = 1$ *and* $\lambda < \pi^2$, *or if* $q > 1$, *problem* (3.2) *has a non-zero solution* U_h.

Proof. The proof of this result is very similar to the standard proof of the existence of a solution of (1.1) when the function $f(u; \lambda)$ grows at a sub-critical rate, and we only sketch it here. The function U_h is a critical point of the functional I defined in (1.5) considered as a map from S_h to R. The Mountain Pass Theorem (for example, [15, Theorem 4.6.1]) then implies that a critical point of $I(U_h)$ exists provided that

(i) $I(0) = 0$ and there exist α, β such that

$$I(u) > 0 \quad \text{if } 0 < |u| < \alpha, \qquad I(v) \geqslant \beta \quad \text{if } |v| = \alpha;$$

(ii) there exists $e \in S_h$ such that $e \neq 0$, $I(e) = 0$;

(iii) if u_n is a sequence in S_h for which $I(u_n) \geqslant 0$ is bounded, and $\Psi(u_n) \to 0$ as $n \to \infty$ (with Ψ defined in (1.7)), then u_n has a convergent subsequence in S_h.

The proof of the geometric conditions (i) and (ii) is as given in [15]. Clearly, for our problem, $I(0) = 0$. Now, for small u, we have that $f(u; \lambda)/u < \pi^2$ and, as S_h is finite-dimensional, the identity map from $H_0^1(S_h)$ to $L^r(S_h)$ is continuous for all r. This, together with the Poincaré inequality, implies that $I(u)$ satisfies (i).

Now let U be fixed in S_h and consider $I(\alpha U)$. As $f(u; \lambda)/u \to \infty$ as $u \to \infty$ then we have that $I(\alpha U) < 0$ for sufficiently large α and hence a function e exists satisfying (ii).

Finally, standard estimates (as in [15]) show that any sequence u_n satisfying the conditions in (iii) must have $\|u_n\|_{H_0^1}$ bounded. As S_h is a finite-dimensional subset of H_0^1, any bounded ball in S_h is compact, and hence, the sequence u_n must have a convergent subsequence.

Combining these results proves the theorem. $\square$

More precise information can be obtained when $f(u; \lambda) = \lambda u + u^p$.

If we set

$$U_h(r) = \sum_{i=0}^{N-1} U_i \varphi_i(r) \tag{3.3}$$

and

$$\mathbf{U} = [\mathbf{U}_0, \ldots, \mathbf{U}_{N-1}]^{\mathrm{T}}, \tag{3.4}$$

then we obtain a system of algebraic equations for the unknowns U_i of the form

$$-\mathbf{AU} + \lambda \mathbf{BU} + \mathbf{G}(\mathbf{U}) = \mathbf{0}, \tag{3.5}$$

where the positive definite, symmetric matrices $\mathbf{A}, \mathbf{B}$ are defined by

$$\mathbf{A}_{ij} = \langle \varphi_i', \varphi_j' \rangle \quad \text{and} \quad \mathbf{B}_{ij} = \langle \varphi_i, \varphi_j \rangle \tag{3.6}$$

and

$$\mathbf{G}(\mathbf{U})_i = \langle U_h^p, \varphi_i \rangle.$$

Elementary bifurcation theory (see [23]) implies that this system has a positive solution branch (such that $U_i > 0$ for all i) which bifurcates from the trivial solution when λ equals the smallest joint eigenvalue $\lambda_{1,h}$ of the matrix pair (A, B), and that this solution branch is unbounded in the sense that $\max[|U_i| + |\lambda|]$ takes arbitrarily large values on the branch.

In [23] it is shown that on such a branch, $\lambda < \lambda_{1,h}$ and $\max(U_i)$ is bounded by inequality (1.16) for all values of λ. Hence the solution branch of the discrete problem must continue up to and beyond the point $\lambda = 0$.

In this case, there are some solution pairs $(\lambda, \mathbf{U})$ on the solution branch for which λ lies in a range for which the solution exists and for which $U_h(r)$ converges to $u(r)$ as $h \to 0$. Conversely, there will be other solution pairs (for example, if $p = 5$ and $\lambda = 0$), which correspond to spurious solutions, and for these $\|U_h\|_\infty$ tends to infinity as $h \to 0$.

The finite-element discretisations of problem (1.1), both for general domains and for symmetric solutions in the sphere, have been considered by several authors, see, for example, the studies in [9, 19,20,23,30]. Furthermore, a review of these works, together with a detailed discussion of the effects of path following, is presented in the monograph of Crouzeix and Rappaz [17]. We can look at the convergence in several norms, however, our interest is primarily in the L_∞ norm, as we have seen in Section 2, that the behaviour of the solutions as a singularity forms can appear to be relatively benign when measured in the H_0^1 or L_2 norms, and significant errors may arise in the discrete solution which are not visible in these norms. The following result on the convergence of the finite-element method in the L_∞ norm is well known:

Theorem 3.1. *Suppose that the function $f(r)$ is continuous and independent of u, and that the function $u(r) \in C^2(0,1)$ satisfies the linear ordinary differential equation*

$$u_{rr} + \frac{2}{r}u_r + f(r) = 0, \quad u_r(0) = u(1) = 0. \tag{3.7}$$

Then, if U_h is a solution of the Galerkin finite-element discretisation of (3.2)*, there is a constant B, independent of u, such that as $h \to 0$ we have*

$$\|u - U_h\|_{L_\infty} \leqslant B\|u_{rr}\|_{L_\infty} h^2 \log(1/h)^\alpha. \tag{3.8}$$

Different authors give different values for α. Dobrowolski and Rannacher [19] give $\alpha = 7/4$, whereas Tourigny [30] gives $\alpha = 1$. This is, however, only a relatively mild change to the rate of convergence.

(The existence of the $\log(1/h)$ term in estimate (3.8) results from estimates associated with a discrete approximation of the Green's function for the Laplacian on the space S_h.)

We now consider the related semi-linear equation

$$u_{rr} + \frac{2}{r}u_r + f(u;\lambda) = 0, \quad u_r(0) = u(1) = 0, \tag{3.9}$$

and make the assumption that u is an *isolated* solution of this equation such that the operator L defined by

$$L\varphi \equiv \varphi_{rr} + \frac{2}{r}\varphi_r + f_u(u;\lambda)\varphi, \quad \varphi_r(0) = \varphi(1) = 0, \tag{3.10}$$

has no non-trivial null eigenvector satisfying $L\varphi = 0$. Provided that the function $f(u)$ is locally Lipshitz-continuous in a neighbourhood of the solution u, we may apply the implicit function theorem to obtain convergence estimates for (3.9) based upon a local linearisation. These estimates extend the result in (3.8) so that a very similar error estimate applies but for which the constant B depends upon the solution u. In particular, B will depend upon $\|L^{-1}\|$. Details of this derivation are given in [17].

For the case of $f(u;\lambda) = \lambda u + u^5$ and large u, we may make the estimate

$$\|u_{rr}\|_\infty \approx \left\|u^5\right\|_\infty = u(0)^5.$$

Hence we have

$$\|u - U_h\|_\infty \leqslant B(u)h^2 \log(1/h)^\alpha u(0)^5. \tag{3.11}$$

It is interesting to compare these results with estimates for convergence in H_0^1. Under the same assumptions of an isolated solution with a locally Lipshitz nonlinearity, we have the well-known result [17]

$$\|u - U_h\|_{H_0^1} < C(u)h.$$

In [23], it is shown that the discrete solution U_h is unique in a ball (in H_0^1) centred upon the projection of u onto the space S_h and of radius R, where

- $R = \mathrm{O}(1)$ if $\varepsilon < 0$,
- $R = \mathrm{O}(h^{\varepsilon/2})$ if $\varepsilon \geqslant 0$.

Thus the region in which U_h can be proven to be unique becomes smaller if $p > 5$ and $h \to 0$, partly accounting for the existence of the spurious solutions described in this paper.

4. Asymptotic calculations

For practical computations, estimate (3.11) for U_h is of limited value. This is for three reasons. Firstly, the value of $B(u)$ can be very large if u is large. Secondly, the estimate is only be descriptive if h is very small. Indeed, for our problem we will demonstrate that (3.11) is descriptive only if $h\|u\|_\infty^4$ is small. We compare this with (1.19) which is descriptive for the much less restrictive range of $H = h\gamma^2$ small. Finally, the analysis of Section 3 gives no information about the behaviour of the spurious solutions.

In this section we use formal methods to obtain instead a descriptive asymptotic description of the solution branch (λ, U_h) for (1.12) which is descriptive provided that U_h is large and H is small. Our methods describe both the convergence and divergence of U_h in the two cases of $\lambda > \pi^2/4$ and $\lambda \leqslant \pi^2/4$.

In Section 2 we noted that, if ε is small and u is large, then close to its maximum (i.e., if r is small) we may approximate the function $u(r)$ by the function $w_\gamma(r)$ for suitable γ. We now make the assumption that such an approximation also holds for the large solutions U_h of the discrete problem, although in this case, we replace the function $w_\gamma(r)$ by its piecewise-linear interpolant $W_{h,\gamma}(r)$ defined in the introduction. We do not aim to prove this assumption, but show that it permits us to make self-consistent asymptotic estimates of the form of the solution which are consistent with all of our numerical computations. Our condition for this approximation to be good is precisely that H is small.

Our method of proof will be to take γ to be large and to compare U_h with $W_{h,\gamma}$ for an *inner* region $0 \leqslant r \ll 1$ using a rescaling of the solution and of r. We then calculate a separate descrition of U_h on an *outer* region which includes $r = 1$. The two representations are then compared in an intermediate (matching) region at $r = 1/\gamma$, allowing a determination of γ.

For convenience, we will look at the two cases of critical growth ($\varepsilon = 0$) and near-critical growth ($\varepsilon \ll 1$) separately.

4.1. Critical growth

The inner region

The discrete solution $U_h \in S_h$ satisfies the equation

$$-\langle U_h', \varphi_h' \rangle + \lambda \langle U_h, \varphi_h \rangle + \langle U_h^5, \varphi_h \rangle = 0 \tag{4.1}$$

for all functions $\varphi_h \in S_h$. Similarly, for all γ and all $\varphi_h \in S_h$, the function w_γ satisfies the equation

$$-\langle w_\gamma', \varphi_h' \rangle + \langle w_\gamma^5, \varphi_h \rangle = 0. \tag{4.2}$$

In contrast, the interpolant $W_{h,\gamma}$ satisfies the equation

$$-\langle W_{h,\gamma}', \varphi_h' \rangle + \langle W_{h,\gamma}^5, \varphi_h \rangle = R\varphi_h, \tag{4.3}$$

where R is a non-zero linear *residual* operator acting on S_h. Setting φ_h equal to the basis function φ_i in (4.3) gives a residual $R_i \equiv R\varphi_i$.

To calculate R_i we rescale the problem, and this rescaling is central to our subsequent analysis. An inspection of (1.26) shows that if

$$v(r) \equiv w_1(r) \tag{4.4}$$

and

$$s = \gamma^2 r, \tag{4.5}$$

then

$$w_\gamma(r) = \gamma v(s) \tag{4.6}$$

and

$$W_{\gamma,h}(r) = \gamma V_H(s), \tag{4.7}$$

where $V_H(s)$ is the piecewise-linear interpolant to $v(s)$ over a uniform mesh of width $H = \gamma^2 h$. Identities (4.2) and (4.3) can now be rescaled by expressing all inner products and derivatives in terms of s. This gives

$$-\langle v', \varphi_H' \rangle + \langle v^5, \varphi_H \rangle = 0 \quad \text{and} \quad -\langle V_H', \varphi_H' \rangle + \langle V_H^5, \varphi_H \rangle = \gamma R\varphi_h, \tag{4.8}$$

where

$$\varphi_H(s) = \varphi_h\left(s/\gamma^2\right).$$

Now, the inner products in (4.8) only depend upon γ indirectly through the scaled variable H and can be compared when H is small. If we set

$$\varphi_{H,i}(s) \equiv \varphi_i\left(s/\gamma^2\right) \tag{4.9}$$

to be a rescaling of the basis function φ_i then subtracting the identities in (4.8) gives

$$-\langle V_H', \varphi_{i,H}' \rangle + \langle V_H^5, \varphi_{i,H} \rangle = -\langle V_H' - v', \varphi_{i,H}' \rangle + \langle V_H^5 - v^5, \varphi_{i,H} \rangle \equiv A_i + B_i. \tag{4.10}$$

As V_H is the interpolant to v on a mesh of size H, the two terms A_i and B_i can be evaluated through an application of standard interpolation theory [16]. This implies that both A_i and B_i are of $\mathrm{O}(H^3)$ as $H \to 0$, and consequently, we may deduce that there exists a function $F(s)$ such that

$$-\langle V_H', \varphi_{i,H}' \rangle + \langle V_H^5, \varphi_{i,H} \rangle = H^3 F(s)\left(1 + \mathrm{O}\left(H^2\right)\right), \tag{4.11}$$

where $s = iH$. Hence, from (4.8),

$$\gamma R_i = H^3 F(s)\left(1 + \mathrm{O}\left(H^2\right)\right),$$

so that

$$R_i = h^3 \gamma^5 F\left(\gamma^2 r\right)\left(1 + \mathrm{O}\left(H^2\right)\right) \tag{4.12}$$

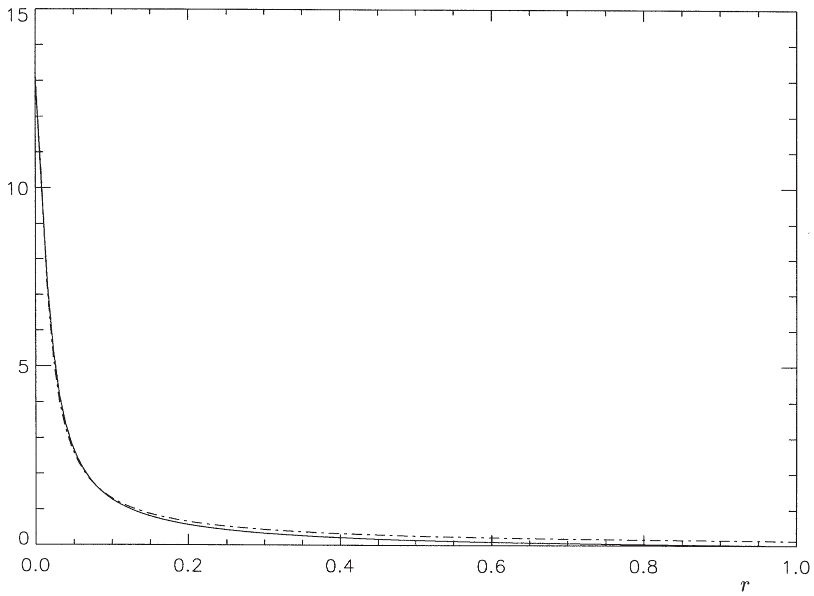

Fig. 2. The function $U_h(r)$ (solid line) compared with $W_{h,\gamma}(r)$ (dashed line).

when

$$r = ih \quad \text{and} \quad s = ih\gamma^2.$$

As both $v(s)$ and $V_H(s)$ are known, the function $F(s)$ may be determined explicitly, and after some manipulation we have

$$F(s) = \frac{\sqrt{3}(8s^6 + 54s^4 - 297s^2 + 54)}{12(3 + s^2)^{9/2}}, \quad s \neq 0, \qquad F(0) = \frac{1}{36}. \tag{4.13}$$

The scaling above introduces both a natural variable s for the inner problem and a natural mesh size H for the rescaled problem. As r approaches 1, then s is large. To match with the outer solution we seek a description of U_h for the case of γ large, s large and r small.

To do this we pose such a solution to be an approximation of $W_{h,\gamma}$ so that

$$U_h = W_{h,\gamma} + e_h, \qquad e_h \ll W_{h,\gamma}, \quad e_h(0) = 0 \quad \text{for } r \ll 1. \tag{4.14}$$

In Fig. 2 we present the results of a calculation of $U_h(r)$ for $h = 1/128$, taking $\lambda = 0$ (where we find that for which $\gamma = 13.1649$) and compare this function with $W_{h,\gamma}(r)$. It is evident from this figure that the difference between U_h and $W_{h,\gamma}$ is indeed small if $r < 0.5$ giving support to our approach.

Substituting expression (4.14) into (4.1) and using (4.3) we then have

$$-\langle e_h', \varphi_h' \rangle + \lambda \langle e_h, \varphi_h \rangle + 5\langle W_{h,\gamma}^4 e_h, \varphi_h \rangle = -\lambda \langle W_{h,\gamma}, \varphi_h \rangle - R\varphi_h + \mathrm{O}(e_h^2). \tag{4.15}$$

For the first stage in the asymptotic calculations, we look at the leading order terms of (4.15), neglecting the terms of $\mathrm{O}(e_h^2)$ to give the linear problem

$$-\langle e_h', \varphi_h' \rangle + \lambda \langle e_h, \varphi_h \rangle + 5\langle W_{h,\gamma}^4 e_h, \varphi_h \rangle = -\lambda \langle W_{h,\gamma}, \varphi_h \rangle - R\varphi_h \quad \forall \varphi_h \in S_h, \tag{4.16}$$

which we solve for e_h. We then estimate the error that neglecting the nonlinear terms introduces to the solution.

Firstly, we rescale (4.16) as a problem in the variable s and then look for solutions when s is large. In the rescaled problem we consider those functions which are members of the set T_H of piecewise-linear functions on the interval $[0,\gamma^2]$, and consistently use the notation $\varphi_H(s)$ for an element of T_H with rescaled basis functions $\varphi_{H,i}(s)$. We now consider all inner products to be integrals with respect to s. Motivated by (4.12), we rescale the residual operator R to a map Q acting on T_H given by

$$R\varphi_h(r) = h^3\gamma^5 Q\varphi_H\big(\gamma^2 r\big) = H^3\gamma^{-1}Q\varphi_H\big(\gamma^2 r\big),$$

so that

$$Q\varphi_H(s) = \frac{1}{h^3\gamma^5}R\varphi_h\big(s/\gamma^2\big).$$

Hence

$$Q\varphi_i \equiv Q\varphi_{H,i}(s) = F(s)\big(1+\mathrm{O}\big(H^2\big)\big), \quad \text{where } s = iH. \tag{4.17}$$

Similarly, we set

$$E_H(s) = e_h\big(\gamma^{-2}s\big).$$

Rescaling the inner products in Eq. (4.16) and multiplying through by γ^2 we obtain

$$\begin{aligned}&-\big\langle E_H', \varphi_H'\big\rangle + \lambda\gamma^{-4}\langle E_H, \varphi_H\rangle + 5\big\langle V_H^4 E_H, \varphi_H\big\rangle = -\lambda\gamma^{-3}\langle V_H, \varphi_H\rangle - H^3\gamma Q\varphi_H \\ &\quad \forall\ \varphi_H \in T_H,\end{aligned} \tag{4.18}$$

where primes now denote differentiation with respect to s.

For large values of s, we have

$$V_H(s) \approx \frac{\sqrt{3}}{s}, \qquad V_H(s)^4 \approx \frac{9}{s^4}.$$

Furthermore, if i is large and $s = iH$, then

$$H^3 Q\varphi_{H,i}(s) \approx \frac{2H^3}{3}\frac{\sqrt{3}}{s^3} \approx H^2\left\langle \frac{2}{3}\frac{\sqrt{3}}{s^5}, \varphi_{H,i}\right\rangle,$$

hence, for such values, most of the terms in (4.18) approximate to zero, and if we make a comparison of the relative sizes of the terms for large s and i, it reduces to the much simpler equation

$$\begin{aligned}&-\big\langle E_H', \varphi_{H,i}'\big\rangle + \frac{\lambda}{\gamma^4}\langle E_H, \varphi_{H,i}\rangle + \frac{\lambda}{\gamma^3}\left\langle \frac{\sqrt{3}}{s}, \varphi_{H,i}\right\rangle \\ &\quad = \mathrm{O}\left(\left\langle \frac{E_H}{s^4}, \varphi_{H,i}\right\rangle\right) + \mathrm{O}\left(H^2\left\langle \frac{1}{s^5}, \varphi_{H,i}\right\rangle\right).\end{aligned} \tag{4.19}$$

Problem (4.19) above is the restriction to the space T_H of the weak form of the ordinary differential equation

$$g'' + \frac{2}{s}g' + \frac{\lambda}{\gamma^4}g + \frac{\lambda}{\gamma^3}\frac{\sqrt{3}}{s} = \mathrm{O}\big(g/s^4\big) + \mathrm{O}\big(H^2/s^5\big), \tag{4.20}$$

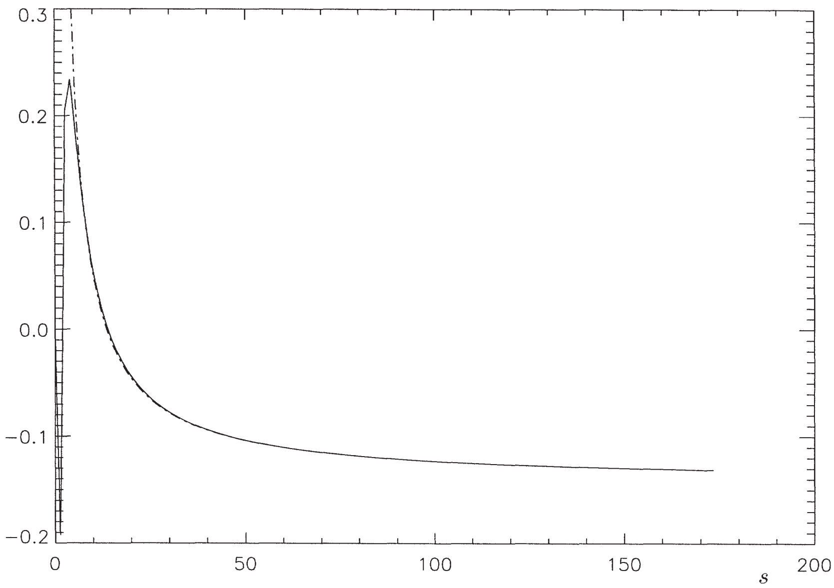

Fig. 3. The error function $E_H(s)$ (solid line) compared with $g(s) = a/s + b$ (dashed line).

which, for large s and small H, has the solution

$$g(s) = \left[\frac{a}{s}\cos\bigl(\lambda^{1/2}\gamma^{-2}s\bigr) + \frac{b}{\lambda^{1/2}\gamma^{-2}s}\sin\bigl(\lambda^{1/2}\gamma^{-2}s\bigr) - \frac{\sqrt{3}\gamma}{s}\bigl(1-\cos\bigl(\lambda^{1/2}\gamma^{-2}s\bigr)\bigr)\right] \times\bigl(1+\mathrm{O}\bigl(1/s^2\bigr)\bigr) \tag{4.21}$$

for constants a and b. Provided that $\lambda^{1/2}\gamma^{-2}s$ is small, we may closely approximate $g(s)$ by

$$g(s) = \left(\frac{a}{s} + b - \frac{\lambda}{\gamma^3}\frac{\sqrt{3}}{2}s\right)\bigl(1+\mathrm{O}\bigl(\lambda s^2\gamma^{-4}\bigr)+\mathrm{O}\bigl(1/s^2\bigr)\bigr). \tag{4.22}$$

For large s, the function $g(s)$ is smooth and standard results from the theory of finite-element approximations imply that the function $E_H(s)$ closely approximates it with a (maximum norm) error proportional to $|H^2 g''(s)| < H^2/s^3$. As $H \ll 1$ and $s \gg 1$, this error is small and we may set

$$E_H(s) = g(s)\bigl(1+\mathrm{O}\bigl(H^2\bigr)\bigr) \tag{4.23}$$

with only a small error. In Fig. 3 we present a graph of the function $E_H(s)$ again computed when $h = 1/128,\ \lambda = 0$, compared with the function $g(s) = a/s + b$, where a and b are estimated (by taking two point values at $s = 0.6\gamma^2,\ s = \gamma^2$) to be

$$a = 1.9383, \qquad b = -0.14273.$$

The agreement between $E_H(s)$ and $g(s)$ is very good for $s > 5$ again lending support to our approximations.

As an interesting observation we note that $E_H(s)$ takes both signs for small values of s contrasting with the continuous problem for which the function $w_\gamma(r) - u(r)$ is always positive.

To estimate b in (4.21) we use integration by parts. To do this we consider the piecewise-linear function

$$\Psi_{h,\gamma}(r) \equiv \frac{\partial W_{h,\gamma}}{\partial \gamma}.$$

Differentiating both sides of the identity

$$R\varphi_i = h^3\gamma^5 F(\gamma^2 r)\big(1 + \mathrm{O}(H^2)\big)$$

with respect to γ we have that in the original unscaled variables

$$-\langle \Psi'_{h,\gamma}, \varphi'_i \rangle + 5\langle W^4_{h,\gamma}, \varphi_i \rangle = \gamma^4 h^3 [5F + 2sF_s]\big(1 + \mathrm{O}(H^2)\big),$$

so that on rescaling we have

$$-\langle \Psi'_H, \varphi'_H \rangle + 5\langle V^4_H \Psi_H, \varphi_H \rangle = H^3 P\varphi_H, \quad \forall \varphi_H \in T_H, \tag{4.24}$$

where

$$\Psi_H(s) \equiv \big(1 - s^2/3\big)\big(1 + s^2/3\big)^{-3/2} \quad \text{if } s = iH, \tag{4.25}$$

is a rescaling of $\Psi_{h,\gamma}(r)$ and P is a linear residual operator, with action on the basis functions given by

$$P_i \equiv P\varphi_{H,i}(s) = (5F + 2sF_s)\big(1 + \mathrm{O}(H^2)\big).$$

The motivation behind these calculations is that Eqs (4.18), (4.24) have a very similar form. To exploit this similarity we define (for $I < N$) piecewise-linear functions $\Psi_I(s), E_I(s) \in T_H$ such that

$$\Psi_I(iH) = \Psi_H(iH), \quad E_I(iH) = E_H(iH), \quad i < I, \qquad E_I(iH) = \Psi_I(iH) = 0, \quad i \geqslant I.$$

As these functions are both in T_H, we may set φ_H to either in identities (4.18) and (4.24). Indeed, setting $\varphi_H = E_I$ in (4.24) and $\varphi_H = \Psi_I$ in (4.18) and subtracting gives

$$\begin{aligned} &-\langle E'_H, \Psi'_I \rangle + \langle \Psi'_H, E'_I \rangle + 5\langle V^4_H E_H, \Psi_I \rangle - 5\langle V^4_H E_I, \Psi_H \rangle \\ &\quad = H^3\gamma Q\Psi_I + \frac{\lambda}{\gamma^3}\langle V_H, \Psi_I \rangle + \frac{\lambda}{\gamma^4}\langle E_H, \Psi_I \rangle - H^3 P E_I. \end{aligned} \tag{4.26}$$

We now examine the two sides of this identity.

On the left-hand side, exploiting the piecewise linearity of the functions in the integrals and the fact that E_H and E_I are identical other than over the interval

$$J \equiv \big[(I-1)H, IH\big],$$

we have, after some manipulation, that

$$\begin{aligned} -\langle E'_H, \Psi'_I \rangle + \langle \Psi'_H, E'_I \rangle &= -\langle E'_H, \Psi'_I - \Psi'_H \rangle + \langle \Psi'_H, E'_I - E'_H \rangle \\ &= \Psi_H(IH)\langle \chi_I, E'_H \rangle - E(IH)\langle \chi_I, \Psi'_H \rangle, \end{aligned} \tag{4.27}$$

where the function $\chi_I(s)$ satisfies

$$\chi_I(s) = \frac{1}{H}, \quad \text{if } s \in J, \quad \text{and} \quad \chi_I(s) = 0, \quad \text{otherwise.}$$

If we take I large so that $s \equiv HI$ satisfies $1 \ll s \ll \gamma^2 \lambda^{-1/2}$, then from (4.21) and (4.23) the terms in this expression can be estimated by

$$E_H(s) = \left(\frac{a}{s} + b - \frac{\lambda\sqrt{3}}{2\gamma^3} s\right)\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\lambda s^2 \gamma^{-4}\right) + \mathrm{O}\left(1/s^2\right)\right), \tag{4.28}$$

$$E'_H(s) = -\left(\frac{a}{s^2} - \frac{\lambda\sqrt{3}}{2\gamma^3}\right)\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\lambda s^2 \gamma^{-4}\right) + \mathrm{O}\left(1/s^2\right)\right), \tag{4.29}$$

and from (4.25),

$$\Psi_H(s) = -\frac{\sqrt{3}}{s}\left(1 + \mathrm{O}\left(1/s^2\right) + \mathrm{O}\left(H^2\right)\right), \qquad \Psi'_H(s) = \frac{\sqrt{3}}{s^2}\left(1 + \mathrm{O}\left(1/s^2\right) + \mathrm{O}\left(H^2\right)\right). \tag{4.30}$$

Using these approximations we have

$$\Psi_H(IH)\left\langle \chi_I, E'_H \right\rangle = \left(\frac{\sqrt{3}a}{HI} + \frac{3HI\lambda}{2\gamma^3}\right)\left(1 + \mathrm{O}\left(\frac{1}{H^2 I^2}\right) + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\lambda I^2 H^2 \gamma^{-4}\right)\right)$$

and

$$E_H(IH)\left\langle \chi_I, \Psi'_H \right\rangle = \left(\frac{\sqrt{3}a}{HI} + \sqrt{3}b - \frac{3HI\lambda}{2\gamma^3}\right)\left(1 + \mathrm{O}\left(\frac{1}{H^2 I^2}\right) + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\lambda I^2 H^2 \gamma^{-4}\right)\right).$$

Thus we estimate (4.27) by

$$\left[\sqrt{3}b - \frac{3HI\lambda}{\gamma^3}\right]\left(1 + \mathrm{O}\left(\frac{1}{H^2 I^2}\right) + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\lambda I^2 H^2 \gamma^{-4}\right)\right) \tag{4.31}$$

and note that, to leading order, the terms in this expression involving the unknown a all cancel.

The remaining terms on the left-hand side of (4.26) give contributions of the form

$$5\left\langle V_H^4 E_H, \Psi_I \right\rangle - 5\left\langle V_H^4 E_I, \Psi_H \right\rangle = \mathrm{O}\left(\int_{(I-1)H}^{IH} V_h^4 E \Psi s^2 \,\mathrm{d}s\right) = \mathrm{O}\left(\frac{1}{H^2 I^2 \gamma^2}\right),$$

which are much smaller than the other expressions in (4.31).

We now consider the right-hand side of (4.26).

The first contribution to this comes from the residual term

$$H^3 \gamma Q \Psi_I.$$

As $\Psi_I \in T_H$, we have

$$\Psi_I(s) = \sum_{i=0}^{I-1} \Psi_i \varphi_{H,i}(s),$$

where $\Psi_i = \Psi(iH)$. Hence, using (4.17),

$$HQ\Psi_I = H\sum_0^{I-1}\Psi_i Q_i = H\sum_0^{I-1}\Psi_H(iH)F(iH)\bigl(1+\mathrm{O}\bigl(H^2\bigr)\bigr). \tag{4.32}$$

Now, both the functions $F(s)$ and $\Psi_H(s)$ decay rapidly with s with product $2/s^4$, and hence, we may approximate this finite sum with an infinite sum with a very small error proportional to $1/s^3 = 1/(IH)^3$. This infinite sum may then in turn be approximated with an additional error of $\mathrm{O}(H)$ by an infinite integral which can be evaluated explicitly using expressions (4.13), (4.25). Thus we have

$$D \equiv H\sum_0^{\infty}\Psi_i Q_i = \left[\int_0^{\infty}\Psi_H(s)F(s)\,\mathrm{d}s\right]\bigl(1+\mathrm{O}(H)\bigr) = \frac{-13\sqrt{3}\pi}{4608}\bigl(1+\mathrm{O}(H)\bigr) \tag{4.33}$$

and

$$H^3\gamma Q\Psi_I = H^2\gamma D\bigl(1+\mathrm{O}\bigl(H^2\bigr)\bigr).$$

The next term in the right-hand side is given by the integral $\lambda\langle V_H,\Psi_I\rangle/\gamma^3$ which is precisely

$$\begin{aligned}&\left[\frac{\lambda}{\gamma^3}\int_0^{IH}\frac{(1-s^2/3)}{(1+s^2/3)^2}s^2\,\mathrm{d}s\right]\bigl(1+\mathrm{O}\bigl(H^2\bigr)\bigr)\\ &\quad=\left[-\frac{3IH\lambda}{\gamma^3}+\frac{3\sqrt{3}\pi\lambda}{\gamma^3}-\frac{27\lambda}{IH\gamma^3}+\mathrm{O}\left(\frac{1}{(IH\gamma)^3}\right)\right]\bigl(1+\mathrm{O}\bigl(H^2\bigr)\bigr).\end{aligned} \tag{4.34}$$

We now consider the remaining two unresolved integrals in the right-hand side of (4.26). In the term $\langle E_H,\Psi_I\rangle$, we estimate E_H by b to give

$$\frac{\lambda}{\gamma^4}\langle E_H,\Psi_I\rangle \approx \frac{\lambda b}{\gamma^4}\int_0^{IH}\frac{s^2(1-s^2/3)}{(1+s^2/3)^{3/2}}\,\mathrm{d}s \approx -\frac{\sqrt{3}b\lambda I^2H^2}{2\gamma^4}.$$

(A more detailed calculation (not presented here) shows that, if we take the complete expression for $E_H(s)$ implied by (4.21) (with the expression $b\sin(\lambda^{1/2}\gamma^{-2}s)/(\lambda^{1/2}\gamma^{-2}s)$ instead of the constant b), then expression (4.31) has an additional term which, to leading order, exactly matches the expression above.)

Finally, estimating E_I by b in the expression H^3PE_I gives

$$H^3PE_I \approx H^2b\sum_0^{I-1}(5F+2sF_s)H \approx H^2b\int_0^{\infty}(5F+2sF_s)\,\mathrm{d}s = \frac{\sqrt{3}}{21}H^2b.$$

Combining all of these estimates gives

$$\begin{aligned}&\left[\sqrt{3}b-\frac{3IH\lambda}{\gamma^3}\right]\left(1+\mathrm{O}\left(\frac{1}{I^2H^2}\right)+\mathrm{O}\bigl(H^2\bigr)+\mathrm{O}\bigl(\lambda I^2H^2\gamma^{-4}\bigr)\right)\\ &\quad=\left[D\gamma H^2-\frac{3IH\lambda}{\gamma^3}+\frac{3\sqrt{3}\pi\lambda}{\gamma^3}-\frac{27\lambda}{IH\gamma^3}\right]\\ &\qquad\times\left(1+\mathrm{O}\left(\frac{1}{I^2H^2}\right)+\mathrm{O}\bigl(H^2\bigr)+\mathrm{O}\bigl(\lambda I^2H^2\gamma^{-4}\bigr)\right).\end{aligned} \tag{4.35}$$

This expression may be regarded as a function of I for large values of I. An inspection reveals that both sides of this identity are fully consistent provided that b takes the value

$$b = \frac{1}{\sqrt{3}}\left[H^2\gamma D + \frac{3\sqrt{3}\pi\lambda}{\gamma^3}\right]\bigl[1 + \mathrm{O}\bigl(H^2\bigr) + \mathrm{O}\bigl(1/I^2H^2\bigr) + \mathrm{O}\bigl(\lambda I^2H^2/\gamma^4\bigr)\bigr]. \tag{4.36}$$

To complete the description of E_h we make an estimate for the magnitude of the term a. A similar such estimate was made in [10] for the continuous problem by obtaining an explicit inversion of the closely related continuous linear operator which corresponds directly to the left-hand side of (4.18). This calculation showed that both a and b had comparable magnitudes. Having established from our calculation of b that the related linear operator in (4.18) has a well-defined inverse we are justified in making a similar estimate for this problem to give

$$a = \mathrm{O}(b) = \mathrm{O}\bigl(\gamma H^2\bigr) + \mathrm{O}\bigl(\gamma^{-3}\bigr). \tag{4.37}$$

Combining (4.22), (4.23), (4.36) and (4.37) thus leads to an estimate of the leading-order terms in E_H.

We now consider the nonlinear terms which were neglected in (4.15) and lead to an error in the estimation of b. The leading contribution to (4.15) arising from these additional terms takes the form

$$10\bigl\langle W_{h,\gamma}^3 e_h^2, \varphi\bigr\rangle; \tag{4.38}$$

after rescaling, we can consider this to be an additional 'forcing' term to Eq. (4.18) of the form

$$\frac{10}{\gamma}\bigl\langle V_H^3 E_H^2, \varphi\bigr\rangle, \tag{4.39}$$

which leads to an additional contribution to the right-hand side of (4.36) of magnitude

$$\frac{10}{\gamma}\bigl\langle V_H^3 E_H^2, \Psi_I\bigr\rangle. \tag{4.40}$$

Estimating the function E_H by b, this term has an approximate magnitude of

$$b^2\frac{10}{\gamma}\int_0^\infty \frac{(1 - s^2/3)}{(1 + s^3/3)^3}s^2\,\mathrm{d}s = b^2\frac{30}{8\gamma}\sqrt{3}\pi. \tag{4.41}$$

The resulting error relative to b is then

$$\frac{b}{\gamma} = \mathrm{O}\bigl(H^2\bigr) + \mathrm{O}\left(\frac{1}{I^2H^2}\right), \tag{4.42}$$

which has the same magnitude as the existing errors. The other nonlinear terms in (4.15) give similar small errors to the calculation of b.

We now combine these results to give a description of $U_h(r)$ in the inner region. To do this we rescale s to r, so that all expressions involving $IH \equiv s$ are replaced by expressions in terms of $\gamma^2 r$. Then, provided that $\gamma^{-2} \ll r \ll \lambda^{-1/2}$, we have

$$\begin{aligned} U_h(r) = W_{h,\gamma}(r) + e_h(r) = \frac{\sqrt{3}}{\gamma r}\left(1 + \frac{3}{\gamma^4 r^2}\right)^{-1/2} - \frac{\sqrt{3}\lambda r}{2\gamma} \\ + \left(\frac{a}{\gamma^2 r} + b\right)\left(1 + \mathrm{O}\bigl(\lambda r^2\bigr) + \mathrm{O}\bigl(H^2\bigr) + \mathrm{O}\left(\frac{1}{\gamma^4 r^2}\right)\right), \end{aligned} \tag{4.43}$$

where

$$b = \frac{1}{\sqrt{3}}\left[\gamma H^2 D + \frac{3\sqrt{3}\pi\lambda}{\gamma^3}\right]\left(1 + \mathrm{O}\left(\lambda r^2\right) + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\frac{1}{\gamma^4 r^2}\right)\right), \tag{4.44}$$

$$a = \mathrm{O}\left(\gamma H^2\right) + \mathrm{O}\left(\gamma^{-3}\right)$$

and

$$D = -\frac{13\sqrt{3}\pi}{4608}\left(1 + \mathrm{O}(H)\right).$$

The outer region

For the second stage in our estimation of γ, we consider the behaviour of U_h in an outer region which includes $r = 1$. In this region, the function $u(r)$ satisfies

$$u_{rr} + \frac{2}{r}u_r + \lambda u + u^5 = 0, \quad u(1) = 0;$$

moreover, $w_\gamma(1) = \sqrt{3}/\gamma$, which is small if γ is large. Consequently, we expect that the solution of both the continuous and discrete problems will be small in a neighbouhood of $r = 1$. Motivated by the scaling for w_γ, we rescale $u(r)$ so that

$$u(r) = \frac{1}{\gamma}v(r),$$

which gives

$$v_{rr} + \frac{2}{r}v_r + \lambda v + \frac{1}{\gamma^4}v^5 = 0, \quad v(1) = 0.$$

A set of solutions of this problem may be determined by, first, solving the linear part of the equation, and then, viewing the nonlinear part as a forcing term. If $\lambda > 0$, this gives (after some manipulation)

$$v(r) = B\left[\frac{\cos(\sqrt{\lambda}r)}{r} - \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r}\right] + \mathrm{O}\left(\frac{1}{\gamma^4 r^3}\right) \tag{4.45}$$

for an arbitrary constant B (with appropriate changes if $\lambda \leqslant 0$). The leading part of this solution will be a good approximation to the whole provided that $1/(\gamma^4 r^3) \ll 1/r$, i.e., if $\gamma^{-2} \ll r$. Thus, if $\gamma^{-2} \ll r \ll \lambda^{-1/2}$, we have (on rescaling)

$$u(r) = \frac{B}{\gamma}\left[\frac{1}{r} - \frac{\lambda r}{2} - \theta(\lambda)\left(1 + \mathrm{O}\left(\lambda r^2\right)\right) + \mathrm{O}\left(r^3\right)\right]\left(1 + \mathrm{O}\left(\frac{1}{\gamma^4 r^3}\right)\right), \tag{4.46}$$

where $\theta(\lambda)$ is given in (1.20). We now consider the weak formulation of (4.46) over the space S_h. As the solution of the original differential equation is smooth if $\gamma^{-2} \ll r$, then it follows from the standard theory of the finite-element method that this weak formulation will have a set of discrete solutions U_h, such that, if U_h and u have the same gradient at $r = 1$, then the error between them will be bounded by

$$\left|h^2 u''\right| \sim \frac{h^2}{\gamma r^3} = \frac{H^2}{\gamma^5 r^3}.$$

Combining these results we deduce that in the outer region and for all B there is a discrete solution U_h of the weak equation satisfying

$$U_h(r) = \frac{B}{\gamma}\left[\frac{1}{r} - \frac{\lambda r}{2} - \theta(\lambda)\bigl(1 + \mathrm{O}\bigl(\lambda r^2\bigr)\bigr) + \mathrm{O}\bigl(r^3\bigr)\right]\left(1 + \mathrm{O}\left(\frac{1+H^2}{\gamma^5 r^3}\right)\right). \tag{4.47}$$

Matching

The two expressions for U_h given in (4.43) and (4.47) are both valid in the range $\gamma^{-2} \ll r \ll \lambda^{-1/2}$ and can be compared there. We see immediately that they have precisely the same functional form in r over this range. Furthermore, they agree quantitatively to leading order provided that

$$\frac{B}{\gamma r} = \frac{\sqrt{3}}{\gamma r} + \frac{a}{\gamma^2 r},$$

so that

$$B = \sqrt{3}\bigl[1 + \mathrm{O}\bigl(H^2\bigr) + \mathrm{O}\bigl(\gamma^{-4}\bigr)\bigr].$$

Comparing the constant terms in the expressions we then have

$$\begin{aligned}
&-\frac{\sqrt{3}\theta(\lambda)}{\gamma}\left(1 + \mathrm{O}\bigl(\lambda r^2\bigr) + \mathrm{O}\bigl(\gamma^{-4}\bigr) + \mathrm{O}\left(\frac{1}{\gamma^4 r^2}\right) + \mathrm{O}\bigl(H^2\bigr)\right)\\
&\quad = b\left(1 + \mathrm{O}\bigl(H^2\bigr) + \mathrm{O}\bigl(\lambda r^2\bigr) + \mathrm{O}\left(\frac{1}{\gamma^4 r^2}\right)\right),
\end{aligned} \tag{4.48}$$

where an expression for b is as given in (4.44). In the above formulae, we have error terms in b and B of the form $\mathrm{O}(\lambda r^2)$ and $\mathrm{O}(1/(\gamma^4 r^2))$. The first of these increases with r, and the second decreases with r, such that they are comparable in magnitude and are of order $\mathrm{O}(1/\gamma^2)$ if we take $r = 1/\gamma$. Hence, we match the expressions at this value of r. Dividing both sides of (4.48) by $\sqrt{3}$ and setting

$$A \equiv -D/3 = \frac{13\pi}{4608\sqrt{3}}\bigl(1 + \mathrm{O}(H)\bigr),$$

we obtain (1.19) in the special case of $\varepsilon = 0$.

Result (1.19) allows us to make some asymptotic estimates about the form of the solution – and, in particular, to calculate the value of γ for different ranges of λ. The form of these changes depending upon the sign of $\theta(\lambda)$. If $\lambda > \pi^2/4$, then $\theta(\lambda) < 0$, γ is bounded above as $h \to 0$ and the corresponding solution U_h converges to the true solution. Conversely, if $\lambda \leqslant \pi^2/4$, then $\theta(\lambda) \geqslant 0$, and γ is unbounded as $h \to 0$.

4.1.1. Case 1: $\lambda > \pi^2/4,\ \|u\|_\infty \gg 1$

When $\lambda > \pi^2/4$, the underlying problem (1.12) has a true solution $u(r)$. It is shown formally in [10] (see also the rigorous results in [8]), that as $\lambda \to \pi^2/4$, then $\|u\|_\infty \to \infty$ with

$$\theta(\lambda) = \frac{-\lambda\sqrt{3}\pi}{\|u\|_\infty^2} + \mathrm{O}\left(\frac{1}{\|u\|_\infty^4}\right). \tag{4.49}$$

Using this relation for θ, we may rewrite (1.19) as

$$\left(Ah^2\gamma^5 - \frac{\lambda\sqrt{3}\pi}{\gamma^3}\right)\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\gamma^{-2}\right)\right) = -\frac{\lambda\sqrt{3}\pi}{\gamma\|u\|_\infty^2}\left(1 + \mathrm{O}\left(\frac{1}{\|u\|_\infty^2}\right)\right). \tag{4.50}$$

Rearranging this gives to leading order (as $\lambda \to \pi^2/4$)

$$\|u\|_\infty = \frac{\gamma}{(1 - Ch^2\gamma^8)^{1/2}}\left(1 + \mathrm{O}\left(H^2\right) + \mathrm{O}\left(\gamma^{-2}\right)\right), \tag{4.51}$$

where

$$C = \frac{4A}{\sqrt{3}\pi^3}.$$

This gives the first result in Formal Proposition 1.2. We may make several deductions from (4.51). First, we note that $\|u\|_\infty$ becomes infinite when $Ch^2\gamma^8 = 1$, and we return to this in the next section.

Secondly, we can use this formula to give a rough estimate for the convergence of U_h to u for small h. If h is very small, such that $h^2\gamma^8 \ll 1$, then expanding (4.51) we have

$$\|u\|_\infty = \gamma + \frac{C}{2}h^2\gamma^9 + \mathrm{O}\left(h^4\gamma^{17}\right). \tag{4.52}$$

For the range $h^2\gamma^8 \ll 1$, this asymptotic formula is comparable with the rigorous estimate (3.11) for the convergence of U_h given in Section 3. Combining these two estimates and using the triangle inequality we have

$$\big|\|u\|_\infty - \|U_h\|_\infty\big| \approx \frac{C}{2}h^2\gamma^9 < \|U_h - u\|_\infty < B(u)\|u\|_\infty^5 h^2 \log(1/h)^\alpha.$$

Due to the existence of the $\log(1/h)$ term, these two estimates are not strictly comparable as $h \to 0$. However, if we have $h\gamma^4 \approx 1$ then γ is close to $\|u\|_\infty$ and

$$B(u) > \frac{C}{8}\frac{\gamma^4}{\log(\gamma)^\alpha},$$

so that $B(u)$ is very large if γ is large. Estimate (4.52) can be used to give a criterion for the effective selection of a computational mesh to solve (1.1) and is discussed further in a forthcoming paper.

For values of h, γ, such that $h^2\gamma^4 \ll 1$, but for which $h^2\gamma^8$ is large, after some manipulation (4.50) simplifies to

$$\gamma \approx C^{-1/8}h^{-1/4},$$

which is independent of u. For this range, formula (4.52) is not descriptive. Hence (4.50) gives a more complete description of the convergence of U_h to u over the range $h^2\gamma^4 \ll 1$ than the standard formula (3.11).

4.1.2. Case 2: $\lambda = \pi^2/4 + O(h^{1/2})$

The value of $\lambda = \pi^2/4$ marks the transition in the behaviour of the analytic solution, and thus we expect to see a change in the behaviour of the discrete solution. This is, indeed, the case – and it is in an $O(h^{1/2})$ neighbourhood of this value that we see a transition between convergent and spurious behaviour.

If we set

$$\lambda = \frac{\pi^2}{4} + \delta$$

and expand $\theta(\lambda)$ as a Taylor series in δ, we obtain

$$\left(A\gamma^5 h^2 - \frac{\sqrt{3}\pi^3}{4\gamma^3}\right)\left(1 + O\left(H^2\right) + O\left(1/\gamma^2\right)\right) = \frac{1}{\gamma}\left(-\frac{\delta}{2}\gamma^2 + O\left(\delta^2\right)\right)\left(1 + O\left(H^2\right) + O\left(\gamma^{-2}\right)\right).$$

Now, introduce new variables

$$\zeta = \gamma h^{1/4}, \qquad \kappa = \delta h^{-1/2}.$$

If κ and ζ are of order one, then all the terms in the above expression have the order γ^{-3}, so that multiplying by γ^3 we have (after some manipulation)

$$\left(C\zeta^8 - 1\right)\left(1 + O\left(h^{1/2}\right)\right) = -\frac{2}{\sqrt{3}\pi^3}\kappa\zeta^2\left(1 + O(h)\right). \tag{4.53}$$

To leading order, (4.53) has a smooth solution path expressing ζ as a function of κ and connects the region of convergent solutions with $\kappa > 0$ to spurious ones with $\kappa < 0$. Thus, in an interval of $O(h^{1/2})$ centred upon $\pi^2/4$, we see solutions for which ζ is of order one and, hence, for which γ grows at the rate $O(h^{-1/4})$. If $\delta > 0$ is fixed and $h \to 0$, then $\kappa \to \infty$, ζ tends to zero and

$$\zeta^{-2} \to \left(\frac{2}{\sqrt{3}\pi^3}\right)\kappa \quad \text{or} \quad \gamma^{-2} \to \left(\frac{2}{\sqrt{3}\pi^3}\right)\delta.$$

In contrast, if $\kappa \to -\infty$, we have (after some manipulation)

$$\zeta^6 \approx -\frac{\kappa A}{2}.$$

Finally, if $\kappa = 0$, we have simply that

$$\zeta^8 = \frac{1}{C}\left(1 + O\left(h^{1/2}\right)\right) \quad \text{or} \quad \gamma = h^{-1/4}C^{-1/8}\left(1 + O\left(h^{1/2}\right)\right).$$

This gives the result (1.23) in Formal Proposition 1.2.

4.1.3. Case 3: $\lambda < \pi^2/4$

Finally, if $\lambda < \pi^2/4$, then $\theta(\lambda) > 0$, and (1.19) admits solutions for arbitrarily large values of γ provided that h is sufficiently small. In particular, as $h \to 0$, we have in this limit

$$Ah^2\gamma^5\left(1 + O\left(H^2\right) + O\left(1/\gamma^2\right)\right) = \frac{1}{\gamma}\theta(\lambda)\left(1 + O\left(H^2\right) + O\left(\gamma^{-2}\right)\right), \tag{4.54}$$

so that

$$\gamma = \left(\frac{\theta(\lambda)}{A}\right)^{1/6} h^{-1/3}\bigl(1 + \mathrm{O}\bigl(h^{2/3}\bigr)\bigr) \tag{4.55}$$

giving the result quoted in (1.24).

4.2. Near-critical growth

We can extend the results of the previous section to consider the case of near-critical growth where the exponent p is close to 5. The calculations in this section are necessarily somewhat more formal than in the last section, but use many of the same ideas, and we do not go into the same detail.

If $p \equiv 5+\varepsilon$ is close to 5, then we may consider the term $u^{5+\varepsilon}$ to be a perturbation of u^5. To leading order, this is given by

$$u^{5+\varepsilon} = u^5 + \varepsilon u^5 \log(u) + \mathrm{O}\bigl(\varepsilon^2 u^5 \log^2(u)\bigr).$$

If we set $U_h = W_{h,\gamma} + e_h$ as before, then to leading order, the linear equation (4.16) has an additional term of the form

$$\varepsilon\gamma^5\left(1 + \frac{\gamma^4 r^2}{3}\right)^{-5/2} \log\left(\gamma\left(1 + \frac{\gamma^4 r^2}{3}\right)^{-1/2}\right)\bigl(1 + \mathrm{O}\bigl(\varepsilon \log^2(\gamma)\bigr)\bigr).$$

Rescaling this equation as a function of s and rearranging gives an additional contribution to the right-hand side of (4.18) of the form

$$-\gamma\varepsilon\langle l, \varphi_h\rangle\bigl(1 + \mathrm{O}\bigl(\varepsilon \log^2(\gamma)\bigr)\bigr),$$

where

$$l(s) \equiv \left(1 + \frac{s^2}{3}\right)^{-5/2}\left[\log(\gamma) - \frac{1}{2}\log\left(1 + \frac{s^2}{3}\right)\right].$$

Now we observe that, if $s \gg 1$, then $l \sim 1/s^5$, which is a negligable contribution to the system, and hence, for large s, $E_h(s)$ takes the same form as in (4.22), although there is an additional contribution to the value of b due to the new terms. This contribution can, as before, be estimated by quadrature to be

$$\left[\frac{\gamma\varepsilon}{\sqrt{3}}\int_0^\infty \Psi_H(s) l(s) s^2 \,\mathrm{d}s\right]\bigl(1 + \mathrm{O}\bigl(\varepsilon \log^2(\gamma)\bigr)\bigr) = -\frac{\gamma\varepsilon\pi}{32}\bigl(1 + \mathrm{O}\bigl(\varepsilon \log^2(\gamma)\bigr)\bigr).$$

We note, with interest, that the contribution to this integral from the terms in $l(s)$ of the form $\log(\gamma)$ vanishes. Combining this result with the existing formula for b extends the leading-order terms in formula (1.19) to

$$Ah^2\gamma^5 - \frac{\sqrt{3}\pi\lambda}{\gamma^3} - \frac{\gamma\varepsilon\pi}{32\sqrt{3}} = \sqrt{\lambda}\cot(\sqrt{\lambda}),$$

which on rearrangement, together with the error estimates above, gives (1.19).

We may draw several new conclusions from this formula.

Firstly, if ε is negative, then γ cannot be unbounded as $h \to 0$ and tends to a limit. This can be determined explicitly from the formula giving the result in Formal Corollary 1.3. Conversely, if ε is positive, then the contribution to the formula from the terms involving λ are of lower order if γ is large. In this case, we see a balance between the terms involving h and ε alone, and the corresponding solution will depend only weakly upon λ. This is confirmed by numerical computations. By balancing these two terms we obtain the estimate for γ, given in Formal Corollary 1.4.

5. Numerical calculations

We now give some numerical evidence to support our conclusions. The algebraic equations (3.6) describing the branch (λ, U_h) can be solved using a path-following algorithm. An initial point on the branch is obtained by performing a Liapunov–Schmidt reduction close to the bifurcation point where the solution is small and using the solution of this problem as a starting point for a solution of the discrete equations. Taking ε fixed, the value of λ is then reduced in small steps until $\lambda = 0$. At the points $\lambda = 0, \pi^2/4$, we also consider continuation in the parameter ε. The resulting nonlinear equations in the system are solved using the Powell hybrid solver SNSQE [26] which performed efficiently and did not use an explicit functional form for the Jacobian of the system.

For all the computations, we consider a system with a uniform mesh of sizes

$$h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, 1/512.$$

Firstly, we give some numerical evidence for the results presented in Formal Proposition 1.2, secondly, we look at the case of $\varepsilon \neq 0$.

5.1. The case $\varepsilon = 0$

In Fig. 1, we presented a combined series of curves showing the bifurcation diagram of the numerical solution for the meshes given above. From these curves, it is evident from the increase of the gradient that interesting behaviour is occuring close to $\lambda = \pi^2/4$. Furthermore, it appears from these graphs that $\|U_h\|_\infty$ increases monotonically as h decreases. For comparison, in Fig. 4, we plot a graph of the leading-order solution γ of Eq. (1.19) with $h = 1/128$ and compare it with the graph of the function $\|U_h\|_\infty$ as a function of λ. The agreement between these two graphs is good both qualitatively and quantitatively, especially considering the approximations made in the asymptotic formulae.

The Pohozaev functional $P(U_h)$, as defined in (2.4), can be determined for any function $U_h \in S_h$. In Fig. 5, we show the values of the scaled function $P(U_h)/h$, expressed as a function of λ for the values of h above. This graph has several interesting features. Firstly, $P(U_h)$ is non-zero for all λ, h. Secondly, it is positive for these values and we make the conjecture that, for the solutions of (1.15), we have

$$P(U_h) > 0.$$

Thirdly, we note that for $\lambda > \lambda_0$, $P(U_h)/h$ decreases monotonically in h and appears to tend to a limit, implying that, to leading order, $P(U_h)$ is proportional to h. In contrast, for $\lambda < \lambda_0$, $P(U_h)/h$ grows as h is reduced. Indeed, $P(U_h)/h$ appears to have a local minimum close to $\lambda = \lambda_0$.

We now make a more quantitative comparison between the numerical results and the asymptotic formulae given in Formal Proposition 1.2.

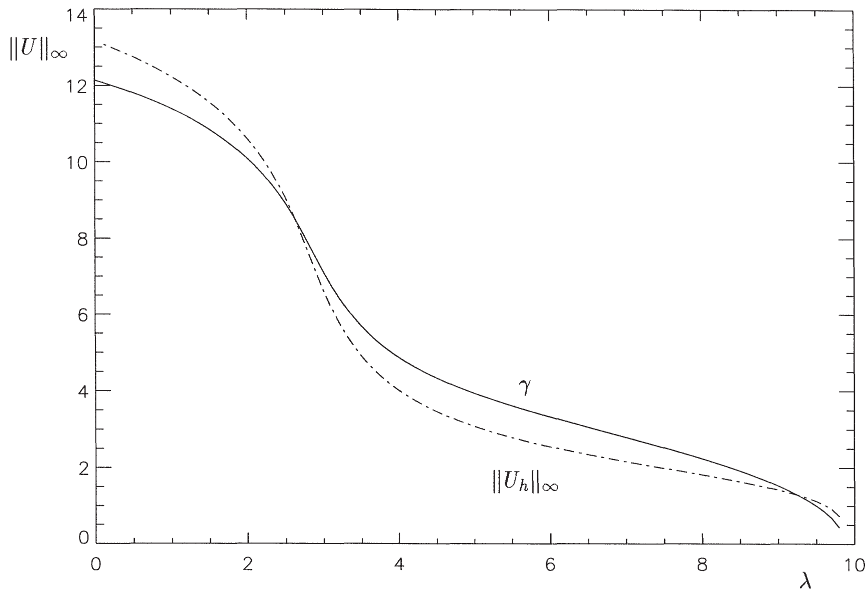

Fig. 4. A comparison of the bifurcation diagram and the asymptotic curve for $h = 1/128$.

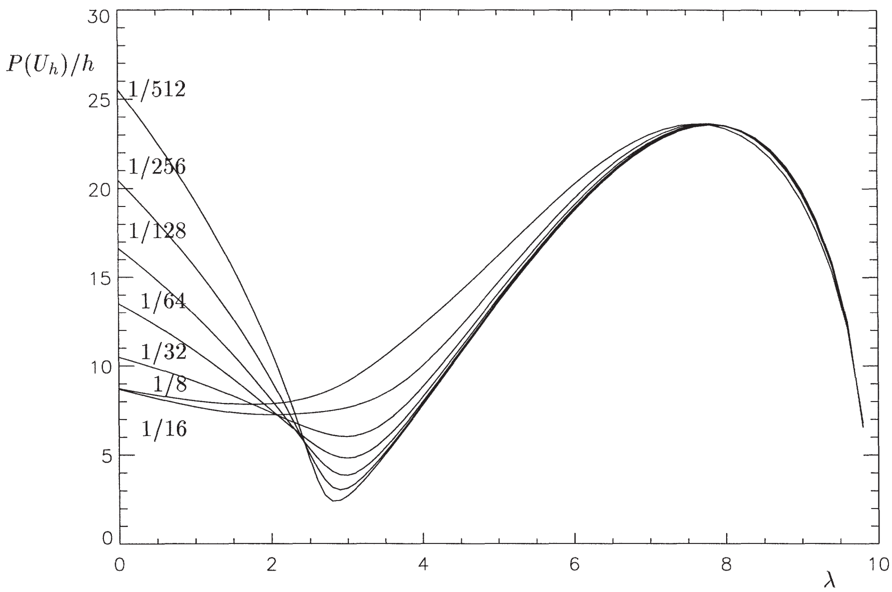

Fig. 5. The scaled Pohozaev functional $P(U_h)/h$ computed for $h = 1/8, \ldots, 1/512$.

5.1.1. Case 1: *Convergence for* $\lambda > \pi^2/4$

For this calculation, we take three values of $\lambda > \pi^2/4$ and compare the solution at each. We have in each case calculated an 'exact' solution by solving (1.12) as an ordinary differential equation using a Runge–Kutta–Mersen method with a very small error tolerence. For these values of λ, such a shooting approach rapidly gave an accurate solution. Table 1 gives the calculated values of $\|u\|_\infty$ and $\gamma = \|U_h\|_\infty$ for $h = 1/512, 1/256, 1/128, 1/64, 1/32$.

Table 1

λ	Exact	$1/512$	$1/256$	$1/128$	$1/64$	$1/32$
2.6	14.0362	11.201	9.871	8.607	7.453	6.388
2.8	8.7897	8.593	8.233	7.626	6.880	6.064
3.0	6.9016	6.879	6.816	6.627	6.254	5.711

Table 2

λ	$1/512$	$1/256$	$1/128$	$1/64$	$1/32$
2.6	0.202	0.297	0.386	0.469	0.544
2.8	0.116	0.063	0.132	0.217	0.310
3.0	0.0033	0.012	0.040	0.094	0.172

Table 3

λ	Exact	$1/512$	$1/256$	$1/128$	$1/64$	$1/32$
2.6	14.0362	11.209	9.834	8.499	7.277	6.193
2.8	8.7897	8.597	8.233	7.586	6.764	5.903
3.0	6.9016	6.87	6.808	6.604	6.180	5.575

Table 4

λ	Exact	$1/512$	$1/256$	$1/128$	$1/64$	$1/32$
2.6	14.0362	14.09	14.28	15.67	24.37	NaN
2.8	8.7897	8.79	8.79	8.89	9.41	10.69
3.0	6.9016	6.90	6.91	6.93	7.09	7.57

From Table 1, we may extract the values for the relative error e_{rel} defined by

$$e_{\text{rel}} = \big(\|u\|_\infty - \gamma\big)/\|u\|_\infty$$

to give Table 2. From this table, it is evident that an asymptotic description of $e_{\text{rel}} \sim Bh^2 \log(1/h)^\alpha$ is not very descriptive if $\lambda = 2.6, 2.8$, although it is reasonable for the finer meshes when $\lambda = 3$.

Rather better understanding of the convergence is given by formula (1.22). We can either use this formula to calculate γ as a function of h (given $\|u\|_\infty$) or to estimate $\|u\|_\infty$ given γ.

For the first case, we solve

$$\|u\|_\infty = \frac{\gamma}{\sqrt{1 - Ch^2\gamma^8}}$$

as a function in γ, for each given value of $\|u\|_\infty$ and h, using the numerical estimate of $C = 3.811 \times 10^{-4}$. This resulting estimates for γ are given in Table 3. These estimates show good agreement with Table 1 lending support to the asymptotic formula for γ in this range.

As the second test for the asymptotic formula, we calculate the quantity U_h^* given by

$$U_h^* = \frac{\gamma}{\sqrt{1 - Ch^2\gamma^8}}.$$

From the asymptotic formula we expect that U_h^* will be a better estimate to $\|u\|_\infty$ than γ. The values of U_h^* are tabulated in Table 4. The agreement between U_h^* and the exact values of $\|u\|_\infty$ for even coarse values of the mesh is impressive.

In Figs 6 and 7, we repeat this calculation to determine a set of extrapolated values $U^*(\lambda)$, with h fixed at $h = 1/128$ and $\lambda \geqslant 2.6$, plotting, respectively, U^* and $(U^*)^{-2}$, and comparing the extrapolated values with $\gamma,\ \|u\|_\infty$ and $\gamma^{-2},\ \|u\|_\infty^{-2}$. In Fig. 6, we see that $\|u\|_\infty,\ \gamma$ and U^* are all close if $\lambda > 3.5$, but that for smaller values of λ, U^* is a much better approximation to $\|u\|_\infty$. The results presented in Fig. 7 make this clear and demonstrate that $(U^*)^{-2} \to 0$ (linearly) as λ approaches λ_0.

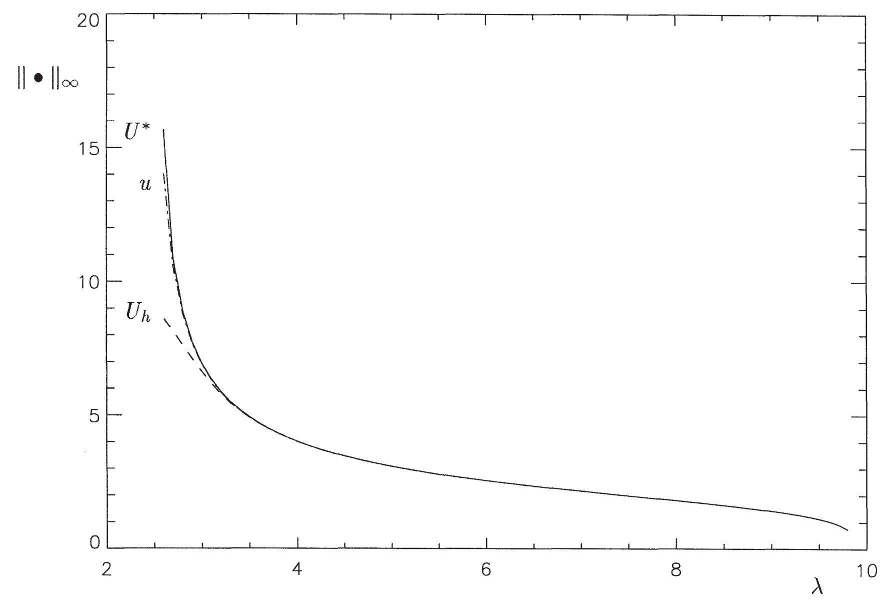

Fig. 6. A comparison of $\|u\|_\infty$, γ and U^*.

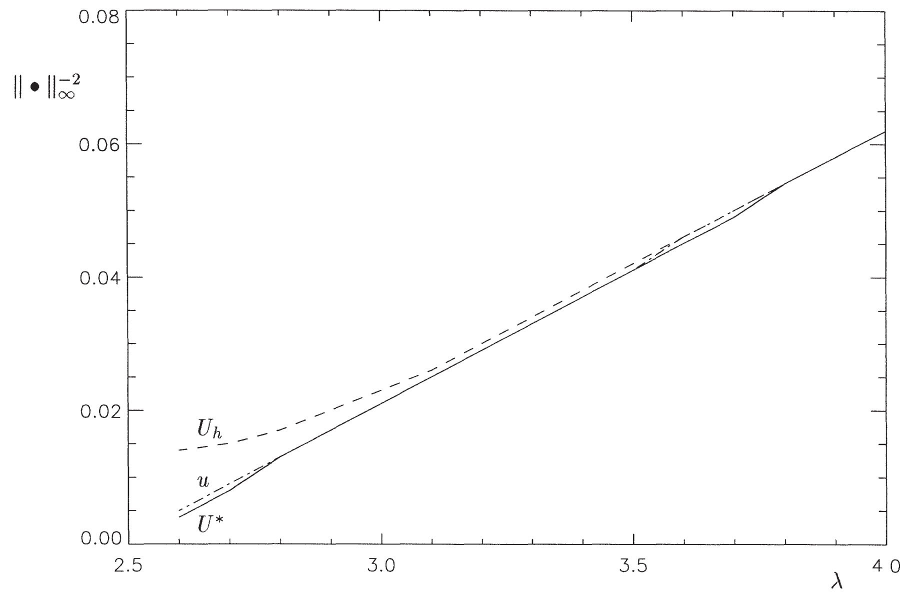

Fig. 7. A comparison of $\|u\|_\infty^{-2}$, γ and $(U^*)^{-2}$.

5.1.2. Case 2: Divergence for $\lambda = \pi^2/4$

We now consider the growth rate of the spurious solutions by first looking at the transitional case. In Table 5, we plot both γ and $h^{1/4}\gamma$. Evaluating the numerical value of the constant $C^{-1/8}$, the asymptotic formula presented in Proposition 1.2 predicts that

$$h^{1/4}\|U_h\|_\infty = 2.6757\big(1 + \mathrm{O}\big(h^{1/2}\big)\big).$$

Table 5

h	γ	$h^{1/4}\gamma$
$1/8$	4.998	2.972
$1/16$	5.704	2.852
$1/32$	6.587	2.769
$1/64$	7.786	2.753
$1/128$	9.172	2.727
$1/256$	10.804	2.701
$1/512$	12.771	2.684

Table 6

h	γ	$h^{1/4}\gamma$	$h^{1/3}\gamma$	$h^{1/2}\gamma$
$1/8$	5.74	3.41	2.87	2.03
$1/16$	7.38	3.69	2.92	1.85
$1/32$	9.08	3.81	2.86	1.61
$1/64$	10.74	3.80	2.69	1.34
$1/128$	13.16	3.91	2.61	1.16
$1/256$	16.38	4.10	2.58	1.02
$1/512$	20.40	4.28	2.55	0.90

The numbers in Table 5 give convincing evidence that this is correct to leading order, and, indeed, that the error term may be an over estimate.

5.1.3. Case 3: Divergence $\lambda = 0$

In this case, we predict that γ will diverge to infinity at a rate proportional to $h^{-1/3}$. Evaluating the numerical value for the constants in the asymptotic formula presented in Formal Proposition 1.2, we now predict that

$$h^{1/3}\gamma = 2.4089\bigl(1 + \mathrm{O}\bigl(h^{2/3}\bigr)\bigr).$$

The values in Table 6 are consistent with this, with fairly good convergence to the predicted constant. In contrast, the values scaled by $h^{1/4}$ appeear to diverge and those scaled by $h^{1/2}$ to tend to zero.

5.2. Varying ε

We now consider the results presented in Corollaries 1.3, 1.4 giving the asymptotic behaviour of the solutions for the exponent values $5+\varepsilon$ when $\lambda = 0$ and $\lambda = \lambda_0$, taking ε to be both positive and negative. To obtain numerical estimates for this problem we solve the system with

$$f(u;\lambda) = \lambda u + u^4, \quad \varepsilon = -1,$$

starting the solution branch as before with $\lambda \approx \pi^2$ and $u \approx 0$. The solution branch is then continued back to $\lambda = \pi^2/4$ and $\lambda = 0$, and at these two (fixed) values of λ we vary the value of ε increasing it to $\varepsilon = 2$.

5.2.1. Case 1: $\lambda = 0$

Let us consider Table 7. Here α is a numerical estimate for the exponent in the relation

$$\gamma = Kh^{-\alpha},$$

Table 7

h	$\varepsilon = -1$	-0.2	0.0	0.2	1	2
$1/32$	6.153	8.151	9.08	9.392	7.605	5.586
$1/64$	6.108	9.192	10.74	12.31	9.947	7.344
$1/128$	6.086	9.895	13.16	14.54	13.06	9.23
$1/256$	6.078	10.18	16.385	18.85	17.19	11.61
α	0	0	0.316	0.374	0.396	0.331

Table 8

h	$\varepsilon = -1$	-0.2	0.0	0.2	1	2
$1/32$	4.42	4.56	6.58	8.01	7.47	5.80
$1/64$	4.427	5.680	7.78	10.47	9.85	7.12
$1/128$	4.418	5.704	9.172	14.229	13.003	8.938
$1/256$	4.418	5.704	10.804	19.657	17.156	11.601
α	0	0	0.236	0.466	0.399	0.33

which is derived from the last two values in the table. The asymptotic predictions for these values are

$$\alpha = 0 \quad \text{if } \varepsilon < 0, \qquad \alpha = 1/3 \quad \text{if } \varepsilon = 0, \quad \text{and} \quad \alpha = 2/(4+\varepsilon) \quad \text{if } \varepsilon > 0.$$

From Table 7, we see that, if $\varepsilon = 1, 2$, then α is close to the predicted values of 0.4 and 0.333. However, the agreement between the asymptotic theory and the results is less good when $\varepsilon = 0.2$.

5.2.2. Case 2: $\lambda = \pi^2/4$

In Table 8, the agreement between the asymptotic theory and the numerical results is good. It is evident, for example, that the exponent α changes from being close to 0.5 when ε is small, to 0.25 when $\varepsilon = 0$.

A second conclusion from the asymptotic calculations is that, if $\varepsilon > 0$, then the computed solution for $\lambda \leqslant \pi^2/4$ should depend only weakly upon λ. This prediction is confirmed by the computations – even for values of ε significantly greater than 0. For example, we see that, if $\varepsilon = 2$ and $h = 1/256$, then the values of γ at $\lambda = 0$ and $\pi^2/4$ are 11.61 and 11.601, respectively.

6. Conclusions

The calculations presented in this paper show both that the behaviour of the finite-dimensional solutions of the elliptic equations close to the critical case is very subtle, and that the asymptotic theory is remarkably accurate in explaining their behaviour. Indeed, it gives very sharp estimates of the growth rate of the spurious solutions and of the convergence properties of the true solutions.

These results are important for more general domains. The estimates presented in [21] and [28] indicate that the singularities, which form in more general domains than the sphere, are locally radially symmetric and close in form to the function w_γ. Hence the finite-element approximation of w_γ and, hence, of these singularities will have similar errors to the calculations presented in this paper. Furthermore, we would expect that the approximation of the solutions away from the singularity will continue to be accurate and goverened by the usual convergence theory described in Section 3, and that this should be descriptive in this case. Thus, it is predicted that similar convergence estimates will be obtained for general domains to the ones in this paper. Some preliminary indications of this for the cube are given in [12]. It remains to be seen whether similar behaviour is obtained in non-convex domains for which the domain shape can also be regarded as a bifurcation parameter. For such domains, there are likely to be ‘critical geometries’ where solutions cease to exist, and we conjecture that for such geometries we will again see spurious solutions growing as $h^{-1/4}$.

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