FLATNESS OF THE NUCLEAR NORM SPHERE, SIMULTANEOUS POLARIZATION, AND UNIQUENESS IN NUCLEAR NORM MINIMIZATION

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ABSTRACT. In this paper we establish necessary and sufficient conditions for the existence of line segments (or *flats*) in the sphere of the nuclear norm via the notion of *simultaneous polarization* and a refined expression for the subdifferential of the nuclear norm. This is then leveraged to provide (point-based) necessary and sufficient conditions for uniqueness of solutions for minimizing the nuclear norm over an affine manifold. We further establish an alternative set of sufficient conditions for uniqueness, based on the interplay of the subdifferential of the nuclear norm and the range of the problem-defining linear operator. Finally, using convex duality, we show how to transfer the uniqueness results for the original problem to a whole class of nuclear norm-regularized minimization problems with a strictly convex fidelity term.

1. INTRODUCTION

One of the most ubiquitous paradigms for linear inverse problems in matrix space is *low rank approximation*, often cast in the form

$$\min_{X\in\mathbb{R}^{n\times p}} \operatorname{rank} X \text{ s.t. } \mathcal{A}(X) = b. \tag{1}$$

Here $\mathcal{A} : \mathbb{R}^{n\times p} \to \mathbb{E}$ is a linear map (into a Euclidean space $\mathbb{E}$) whose action is often simply a matrix multiplication $\mathcal{A}(X) = A \cdot X$ for some $A \in \mathbb{R}^{m\times n}$ or a selection operator which projects X onto the matrix composed of its entries from a prescribed index set $J \subset \{1, \ldots, n\} \times \{1, \ldots, p\}$. We direct the interested reader to Fazel's thesis [4], the important paper by Candès and Recht [2] as well as the survey article by Recht et al. [15] for applications, solution methods and pointers to the abundant literature for the low rank minimization problem (1) and the low rank minimization paradigm in general.

Due to the combinatorial nature of the rank function, problem (1) is, generally, NP-hard (as it contains cardinality minimization as a special case, which is NP-hard [5, 14]), and therefore many continuous relaxations for its numerical solution have been proposed. The predominant class of convex relaxations uses the *nuclear norm* (or *trace norm*) $\|\cdot\|_*$ as a convex approximation of the rank function. The justification for this stems from the fact that the nuclear norm is the *convex envelope* (i.e. the largest convex minorant) of the rank function when restricted to a spectral norm ball around the point in question, a fact that was first established by Fazel in

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her thesis [4] (see also the approach by Hiriart-Urruty and Len [7]). On the other hand, the nuclear norm is simply the ℓ_1-norm of the vector of singular values, and the ℓ_1-norm is known to promote sparsity [3], hence the nuclear norm promotes low rank. Various nuclear norm-based approximations of problem (1) have been proposed, the most obvious one being

$$\min_{X\in\mathbb{R}^{n\times p}} \|X\|_* \text{ s.t. } \mathcal{A}(X) = b. \tag{2}$$

Existence of solutions for this problem[1] is readily established as the objective function is *coercive* (and the suitable continuity properties are satisfied). Given a solution $\bar{X}$ of (2), the goal of this paper is to establish conditions that guarantee that $\bar{X}$ is, in fact, the unique solution. This is inspired by the study by Zhang et al. [25] which establishes uniqueness results for ℓ_1-minimization problems[2]

We approach this task by combining tools from convex analysis and linear algebra. The natural interplay of these areas is most obvious in the study of *unitarily invariant norms* [9] which comes into play here since the nuclear norm (and its dual norm, the spectral norm) are unitarily invariant. This theory goes back to work of von Neumann's [13], expanded on by various authors including Watson [21, 22], Zietak [23, 24] and de Sá [18, 19], and then vastly generalized beyond norms in Lewis' seminal work [10, 11, 12].

Contributions. Our first main contribution, Theorem 3.4, provides a characterization of the existence of line segments (*flatness*) in the boundary of the nuclear norm ball, based on the notion of *simultaneous polarizability* (Definition 3.1). In Corollary 3.6 we give a reformulation of this characterization using the singular value decomposition of a point in the nuclear norm sphere, and this directly carries over to *necessary and sufficient* conditions for uniqueness (Corollary 4.1) for solutions of the nuclear norm minimization problem (2).

We then extend the study by Zhang et al. [25] to the nuclear norm setting, starting from the following observation of Gilbert's [6] for any (proper) convex function f (see Proposition 4.2): $\bar{x}$ is the unique minimizer of f if 0 is in the interior of the subdifferential of f at $\bar{x}$. We make these conditions concrete for problem (2) in Proposition 4.4. We then bridge between these convex-analytic conditions and the linear-algebraic ones established earlier in Corollary 4.1 explicitly in Proposition 4.7, thus illuminating their connection. By means of a counterexample (Example 4.8) we show that the sufficient conditions (Assumption 4.3) are not necessary for uniqueness, which is in contrast to the (polyhedral convex) ℓ_1-case.

Through convex analysis (Proposition 4.9) we are able to transfer our findings for problem (2) to another class of nuclear norm minimization problems (see Corollary 4.10) including nuclear norm-regularized least-squares.

Roadmap. We present in Section 2 the necessary background from linear algebra and convex analysis, including a novel result on the convex geometry of the subdifferential of the nuclear norm. Section 3 is devoted to characterizing the existence of line segments in the nuclear norm sphere. We transfer these findings to nuclear norm minimization problems in Section 4. We close out with some final remarks in Section 5.

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[1] Of course, we assume throughout that this problem is feasible.

[2] Nuclear norm minimization contains ℓ_1-minimization as a special case since $x \in \mathbb{R}^n$ can be identified with a diagonal matrix $\mathrm{diag}(x)$ for which $\|\mathrm{diag}(x)\|_* = \|x\|_1$.

Notation. The vector $e_i \in \mathbb{R}^n$ is the i-th standard unit vector in $\mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, $\operatorname{diag}(x)$ will be a diagonal matrix with x on its diagonal, whose size will be clear from the context (and which may be rectangular). For $X \in \mathbb{R}^{n\times p}$, we will generate the vector of its diagonal entries via $\operatorname{DIAG}(X)$. The space of $n \times n$ (real) symmetric matrices is denoted by $\mathbb{S}^n$, $\mathbb{S}^n_+$ is the positive semidefinite cone while $\mathbb{S}^n_{++}$ denotes the positive definite matrices in $\mathbb{S}^n$. The set of $n \times n$ orthogonal matrices is denoted by $O(n)$. For a set C in a real vector space, we define $\mathbb{R}_+C := \{tx \mid t \geqslant 0,\ x \in C\}$, the smallest cone that contains C. The line segment between two points x, y in a real vector space is denoted by $[x, y]$. The set of all linear maps between two Euclidean spaces V, W is denoted by $\mathcal{L}(V, W)$. For $\mathcal{A} \in \mathcal{L}(V, W)$, we write $\ker \mathcal{A}$ and $\operatorname{rge} \mathcal{A}$ for its *kernel* and *range*, respectively. Its adjoint map is denoted by $\mathcal{A}^*$.

2. Preliminaries

In what follows, $\mathbb{E}$ will be a Euclidean space, i.e. a finite-dimensional real inner product space with its ambient inner product denoted by $\langle\cdot, \cdot\rangle$. The induced norm is denoted by $\|\cdot\|$, i.e. $\|x\| := \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{E}$. For instance, we equip $\mathbb{R}^{n\times p}$ with the (Frobenius) inner product

$$\langle X, Y\rangle := \operatorname{tr}(X^TY) \quad \forall X, Y \in \mathbb{R}^{n\times p},$$

which induces the *Frobenius norm*

$$\|X\| := \sqrt{\langle X, X\rangle} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ij}^2} \quad \forall X \in \mathbb{R}^{n\times p}.$$

For $X \in \mathbb{R}^{n\times p}$ its *nuclear norm* is given by

$$\|X\|_* := \operatorname{tr}(\sqrt{X^TX}) = \operatorname{tr}(\sqrt{XX^T}).$$

The definition implies the following fact used frequently in our study:

$$\|X\|_* = \operatorname{tr}(X) \quad \forall X \in \mathbb{S}^n_+. \tag{3}$$

The *dual norm* of the nuclear norm is

$$\|X\|_{op} := \max_{\|Y\|_*\leqslant 1} \langle X, Y\rangle = \max_{\|v\|\leqslant 1} \|Xv\|,$$

which is called the *operator norm* or *spectral norm*. In what follows, we will define

$$\mathbb{B}_{op} := \{X \mid \|X\|_{op} \leqslant 1\}$$

to be the operator norm unit ball in a matrix space whose dimension will be clear from the context. The following simple estimate for the operator norm will be useful for our study.

Lemma 2.1. *For $A \in \mathbb{R}^{n\times p}$. Then the Euclidean norm of every column and row of A is bounded above by $\|A\|_{op}$. In particular, we have $a_{ij} \leqslant \|A\|_{op}$ for all $i = 1, \ldots, n,\ j = 1, \ldots, p$.*

Proof. Let a_j be the j-th column of A. Then

$$\|a_j\| = \|Ae_j\| \leqslant \sup_{\|x\|=1} \|Ax\| = \|A\|_{op}.$$

Multiplying standard unit vectors e_j^T from the left, we get the analogous statement for rows. □

The following estimate for the nuclear norm of block matrices is important to our study.

Lemma 2.2. *Let $n > p$, $X \in \mathbb{R}^{n\times p}$ and $Y \in \mathbb{R}^{n\times(n-p)}$. Then*

$$\|X\|_* \leqslant \|[X\ Y]\|_*,$$

where equality holds if and only if $Y = 0$.

Proof. Observe that $\|W\|_{op} = \|[W\ 0]\|_{op}$. Hence

$$\mathcal{C} := \left\{[W\ 0] \in \mathbb{R}^{n\times n} \;\middle|\; W \in \mathbb{R}^{n\times p},\ \|W\|_{op} \leqslant 1\right\} \subset \mathbb{B}_{op}.$$

Consequently

$$\|[X\ Y]\|_* = \max_{[W\ Z]\in\mathbb{B}_{op}} \langle [X\ Y],\, [W\ Z]\rangle \geqslant \max_{[W\ 0]\in\mathcal{C}} \langle [X\ Y],\, [W\ 0]\rangle = \|X\|_*.$$

Clearly, the inequality is strict if $Y \neq 0$ (use, e.g., $Z = Y$) and an equality otherwise. □

We point out that the above result allows one to always embed problem (2) (defined by $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n\times p}, \mathbb{E})$ and $b \in \mathbb{E}$) in (potentially rectangular) matrix space $\mathbb{R}^{n\times p}$ (w.l.o.g. $n \geqslant p$) into the (square) matrix space $\mathbb{R}^{n\times n}$. To this end, identify every element $\tilde{X} \in \mathbb{R}^{n\times n}$ with the block matrix $\tilde{X} = [X\ Y]$ for $X \in \mathbb{R}^{n\times p}, Y \in \mathbb{R}^{n\times(n-p)}$, define the linear operator $\tilde{\mathcal{A}} : \tilde{X} \to \mathcal{A}(X)$ and the right-hand side $\tilde{b} := b$. If we now consider the 'padded' problem

$$\min_{[X\ Y]\in\mathbb{R}^{n\times n}} \|[X\ Y]\|_* \text{ s.t. } \tilde{\mathcal{A}}([X\ Y]) = \tilde{b} \tag{4}$$

it is an immediate consequence of Lemma 2.2 that $\bar{X}$ is a solution of (2) if and only if $[\bar{X}\ 0]$ is a solution of (4).

Singular value decomposition. For the facts and concepts presented in this paragraph we refer the uninitiated reader to Horn and Johnson [9] for details. Throughout (w.l.o.g.) we assume that $n \geqslant p$. For $X \in \mathbb{R}^{n\times p}$, with $\operatorname{rank} X = r$, there exist orthogonal matrices $U \in O(n)$ and $V \in O(p)$ (with columns $u_1, \dots, u_n$ and $v_1, \dots, v_p$, respectively) and unique real numbers

$$\sigma_1(X) \geqslant \sigma_2(X) \geqslant \sigma_r(X) > 0 = \sigma_{r+1} = \cdots = \sigma_n(X)$$

such that

$$X = U\operatorname{diag}(\sigma(X))V^T = \sum_{i=1}^r \sigma_i(X) u_i v_i^T.$$

This is called a *singular value decomposition* (SVD) of X. Note that the positive singular values of X are exactly the square roots of the nonzero eigenvalues of XX^T (or X^TX). We say that two matrices $X, Y \in \mathbb{R}^{n\times p}$ have a *simultaneous singular value decomposition* if there exist $(\bar{U}, \bar{V}) \in O(n) \times O(p)$ such that

$$X = \bar{U}\operatorname{diag}(\sigma(X))\bar{V}^T \quad\text{and}\quad Y = \bar{U}\operatorname{diag}(\sigma(Y))\bar{V}^T.$$

The next result, see e.g. [10, Theorem 2.1], due to von Neumann, characterizes simultaneous singular value decompositions.

Theorem 2.3 (von Neumann)**.** *For $X, Y \in \mathbb{R}^{n\times p}$ we have*

$$\langle X,\, Y\rangle \leqslant \langle \sigma(X),\, \sigma(Y)\rangle.$$

Equality holds if and only if X and Y have simultaneous singular value decompositions.

Through the singular value decomposition, we generate the map

$$\sigma : X \in \mathbb{R}^{n\times p} \to \sigma(X) \in \mathbb{R}^p.$$

Using this, the nuclear and operator norm of X, respectively, can be expressed as the ℓ_1- and ℓ_∞-norm, respectively, of the vector of singular values of X, i.e.

$$\|X\|_* = \sum_{i=1}^r \sigma_i(X) \quad \text{and} \quad \|X\|_{op} = \sigma_1(X).$$

Moreover, we find that the nuclear and the operator norm are *orthogonally invariant*, i.e. for all $X \in \mathbb{R}^{n\times p}$, we have

$$\|UXV\|_* = \|X\|_* \quad \text{and} \quad \|UXV\|_{op} = \|VXU\|_{op} \quad \forall (U,V) \in O(n) \times O(p). \tag{5}$$

There is an important extension of the above equation in the rectangular case. To formulate it, we recall the *Stiefel manifold* [20].

Definition 2.4 (Stiefel manifold)**.** *The* Stiefel manifold $\mathcal{V}_{n,p}$ *is the collection of matrices in* $\mathbb{R}^{n\times p}$ *with orthonormal columns, i.e.*

$$\mathcal{V}_{n,p} := \left\{ U \in \mathbb{R}^{n\times p} \;\middle|\; U^T U = I_p \right\}.$$

The nuclear norm also has invariance on one side by multiplication by elements of the Stiefel manifold.

Lemma 2.5. *Let* $X \in \mathbb{R}^{n\times p}$ *and* $U \in \mathcal{V}_{n,p}$. *Then* $\|XU^T\|_* = \|X\|_*$.

Proof. As U has orthonormal columns, we may extend it to an orthogonal matrix $[U\ W] \in O(n)$. Then

$$\|XU^T\|_* = \left\| [X\ 0] \cdot \begin{bmatrix} U^T \\ W^T \end{bmatrix} \right\|_* = \|[X\ 0]\|_* = \|X\|_*,$$

where the second identity uses the orthogonal invariance from (5) and the third is due to Lemma 2.2. □

Tools from convex analysis. For the facts and concepts presented in this paragraph we refer the uninitiated reader to the textbooks by Rockafellar [16], Hiriart-Urruty and Lemaréchal [8], Borwein and Lewis [1] or Rockafellar and Wets [17, Chapter 11].

A function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is called proper if $\operatorname{dom} f := \{x \mid f(x) < +\infty\} \neq \emptyset$. We say that f is *convex* if its epigraph $\operatorname{epi} f := \{(x,\alpha) \in \mathbb{E} \times \mathbb{R} \mid f(x) \leqslant \alpha\}$ is convex, and we say that it is *closed* if $\operatorname{epi} f$ is closed. Its (Fenchel) conjugate $f^* : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ is $f^*(y) := \sup_{x\in \operatorname{dom} f}\{\langle y,\, x\rangle - f(x)\}$. Its (convex) subdifferential at $\bar{x} \in \operatorname{dom} f$ is given by

$$\partial f(\bar{x}) := \{y \in \mathbb{E} \mid f(\bar{x}) + \langle y,\, x - \bar{x}\rangle \leqslant f(x)\ \forall x \in \operatorname{dom} f\}.$$

An important (proper, convex) extended real-valued function is the *indicator function* of a (nonempty, convex) set $C \subset \mathbb{E}$ which is

$$\delta_C : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}, \quad \delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & \text{else.} \end{cases}$$

Its subdifferential is $\partial\delta_C(\bar{x}) = \{v \mid \langle v,\, x - \bar{x}\rangle \leqslant 0\ \forall x \in C\}$ for all $\bar{x} \in C$. Its conjugate is the support function of C, i.e. $\delta_C^*(y) = \sup_{x\in C}\langle x,\, y\rangle =: \sigma_C(y)$. We point out that the support functions of compact, convex, symmetric sets C that contain 0 (thus $0 \in \operatorname{int} C$) are exactly the norms on $\mathbb{E}$ [16, Theorem 15.2] .

The subdifferential of the ℓ_1-norm $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R},\ \|x\|_1 = \sum_{i=1}|x_i|$, reads

$$\partial\|\cdot\|_1(x) = \left(\begin{cases}\operatorname{sgn}(x_i), & x_i \neq 0\\ [-1,1], & x_i = 0\end{cases}\right)_{i=1}^n = \{y \in \mathbb{B}_\infty \mid \langle x, y\rangle = \|x\|_1\}. \tag{6}$$

Obviously, the most important example to our study is the subdifferential of the nuclear norm.

Proposition 2.6 (Subdifferential of nuclear norm). *Let $\bar{X} \in \mathbb{R}^{n\times p}$ and let $(\bar{U}, \bar{V}) \in O(n) \times O(p)$ such that*

$$\bar{U}\operatorname{diag}(\sigma(\bar{X}))\bar{V}^T = \bar{X}.$$

The following hold:

(a) *We have $Y \in \partial\|\cdot\|_*(\bar{X})$ if and only if $\bar{X}$ and Y have a simultaneous singular value decomposition and $\sigma(Y) \in \partial\|\cdot\|_1(\sigma(\bar{X}))$.*

(b) *It holds that*

$$\partial\|\cdot\|_*(\bar{X}) = \left\{Y \;\middle|\; \left\langle \bar{X}, Y\right\rangle = \|\bar{X}\|_*,\ \|Y\|_{op} \leqslant 1\right\}^3 \tag{7}$$

$$= \bar{U}\partial\|\cdot\|_*(\operatorname{diag}(\sigma(\bar{X}))\bar{V}^T. \tag{8}$$

Proof. (a) See [10, Corollary 2.5].

(b) The expressions for the subdifferential can be found in [24], see, in particular, [24, Theorem 3.1] for the characterization in (8). □

For a convex set $C \subset \mathbb{E}$, its *affine hull*, denoted by $\operatorname{aff} C$, is the smallest affine set that contains C. In particular, $\operatorname{aff} C$ is a subspace if and only if it contains 0. The *subspace parallel to* C is defined to be the unique subspace parallel to $\operatorname{aff} C$ and given by $\operatorname{par} C := \operatorname{aff} C - \bar{x}$ for any $\bar{x} \in C$. Clearly, this entails that $\operatorname{ri} C = \operatorname{int} C$ if and only if the latter is nonempty, i.e. when $\operatorname{par} C = \mathbb{E}$.

The *relative interior* $\operatorname{ri} C$ of C is its interior in the relative topology with respect to its affine hull. The following characterization of relative interior points is useful to our study, see, e.g., [1, Exerc. 13, Ch. 1]:

$$x \in \operatorname{ri} C \quad\Longleftrightarrow\quad \mathbb{R}_+(C - x) = \operatorname{par} C. \tag{9}$$

For more details on the relative interior we refer the reader to Rockafellar [16, Chapter 6].

We will now exploit the representation in (8) to derive yet another representation of the subdifferential of the nuclear norm as well as its relative interior and parallel subspace. This is useful to our study but also of independent interest. We need the following lemma.

Lemma 2.7. *For $r \leqslant p (\leqslant n)$ set*

$$\mathcal{T} := \left\{B \in \mathbb{R}^{n\times p} \;\middle|\; \operatorname{DIAG}(B) \in \{1\}^r \times \mathbb{R}^{n-r},\ \|B\|_{op} \leqslant 1\right\}.$$

Then

$$\mathcal{T} = \left\{\begin{pmatrix} I_r & 0\\ 0 & R\end{pmatrix} \;\middle|\; R \in \mathbb{R}^{(n-r)\times(p-r)},\ \|R\|_{op} \leqslant 1\right\}.$$

Proof. Let $B \in \mathcal{T}$. Then, by Lemma 2.1 and the fact that $b_{ii} = 1$ for all $i = 1, \dots, r$, we find that $B = \begin{pmatrix} I_r & 0\\ 0 & R\end{pmatrix}$ for some $R \in \mathbb{R}^{(n-r)\times(p-r)}$. Now observe that

$$\|R\|_{op} \leqslant 1 \quad\Longleftrightarrow\quad \left\|\begin{pmatrix} I_r & 0\\ 0 & R\end{pmatrix}\right\|_{op} \leqslant 1.$$

This shows the desired equality. □

Proposition 2.8 (Convex geometry of $\partial\|\cdot\|_*(\bar X)$)**.** *Let* $\bar X \in \mathbb{R}^{n\times p}$ *with* $r := \operatorname{rank} \bar X$ *and let* $(\bar U, \bar V) \in O(n)\times O(p)$ *such that*

$$\bar U \operatorname{diag}(\sigma(\bar X))\bar V^T = \bar X.$$

Then the following hold:

(a) $\partial\|\cdot\|_*(\bar X) = \bar U \left\{ \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \,\middle|\, R \in \mathbb{R}^{(n-r)\times(p-r)},\ \|R\|_{op} \leqslant 1 \right\} \bar V^T$;

(b) $\operatorname{ri}(\partial\|\cdot\|_*(\bar X)) = \bar U \left\{ \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \,\middle|\, R \in \mathbb{R}^{(n-r)\times(p-r)},\ \|R\|_{op} < 1 \right\} \bar V^T$*;*

(c) $\operatorname{par}(\partial\|\cdot\|_*(\bar X)) = \bar U \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & R\end{smallmatrix}\right) \,\middle|\, R \in \mathbb{R}^{(n-r)\times(p-r)} \right\} \bar V^T$*.*

Proof. (a) From the characterization (8) we find that $\bar U \partial\|\cdot\|_*(\operatorname{diag}(\sigma(\bar X))\bar V^T$. In turn, by (7), we find that

$$\partial\|\cdot\|_*(\operatorname{diag}(\sigma(\bar X)) = \left\{ B \in \mathbb{R}^{n\times p} \,\middle|\, \left\langle B, \operatorname{diag}(\sigma(\bar X)\right\rangle = \|\operatorname{diag}(\sigma(\bar X))\|_*,\ \|B\|_{op} \leqslant 1 \right\}.$$

Now, observe that, by taking adjoints, $\left\langle B, \operatorname{diag}(\sigma(\bar X)\right\rangle = \langle \operatorname{DIAG}(B), \sigma(\bar x)\rangle$, and also $\|\operatorname{diag}(\sigma(\bar X))\|_* = \|\sigma(\bar X)\|_1$. Hence

$$\partial\|\cdot\|_*(\operatorname{diag}(\sigma(\bar X)) = \left\{ B \in \mathbb{R}^{n\times p} \,\middle|\, \operatorname{DIAG}(B) \in \{1\}^r \times \mathbb{R}^{n-r},\ \|B\|_{op} \leqslant 1 \right\},$$

and thus Lemma 2.7 gives the desired result.

(b) Define $F : \mathbb{R}^{(n-r)\times(p-r)} \to \mathbb{R}^{n\times n}$ by $F(R) = \bar U \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \bar V^T$. Then, in view of (a), we find that $\partial\|\cdot\|_*(\bar X) = F(\mathbb{B}_{op})$. Therefore the desired formula follows from [16, Theorem 6.6].

(c) Follows immediately from (a) or (b). □

In the setting of Proposition 2.8, it follows immediately from part (c) that

$$\operatorname{par}(\partial\|\cdot\|_*(\bar X)) = \operatorname{span}\left\{ u_i v_j^T \mid i = r+1,\dots,n,\ j = r+1,\dots,p \right\}, \tag{10}$$

where u_i $(i = 1,\dots,n)$ and v_j $(j = 1,\dots,p)$ are the columns of $\bar U$ and $\bar V$, respectively.

3. Flatness of the nuclear norm and simultaneous polarizability

As before, we assume (w.l.o.g.) that $n \geqslant p$. In this section we present our main results on the geometry of the nuclear norm sphere, specifically a characterization of the *flats*[4]. We then leverage this to characterize the uniqueness of certain nuclear norm optimization problems. The next definition is central to this analysis.

Definition 3.1 (Polarizability)**.** *Let* $X, \hat X \in \mathbb{R}^{n\times p}$*.*

(a) *We say that* $U \in \mathcal{V}_{n,p}$ polarizes[5] X *if* $XU^T \in \mathbb{S}^n_+$*.*

(b) *We say that* X *and* $\hat X$ *are* simultaneously polarizable *if there exists a matrix* $U \in \mathcal{V}_{n,p}$ *that polarizes both* X *and* $\hat X$*.*

A polarization in the sense of Definition 3.1 (a) always exists as the following result shows, which is based on *polar decomposition.* Note that conventionally, for the case of the rectangular polar decomposition, the polarizing matrix U usually appears on the small side of the matrix, see Horn and Johnson [9, Theorem 7.3.1]. In Definition 3.1, we have placed it on the large side, but we observe that by padding, it is possible to conclude the existence of the large polarization as well.

[4]Flats, in the context of Riemannian geometry, are (uncurved) Euclidean submanifolds.

[5]Sometimes this is also called the ‘angular’ part of the polar decomposition.

Proposition 3.2 (Existence of polarization)**.** *Let $X \in \mathbb{R}^{n\times p}$. Then there exists $U \in \mathcal{V}_{n,p}$ that polarizes X.*

Proof. Consider the augmented matrix $[X\ 0] \in \mathbb{R}^{n\times n}$. By polar decomposition, see, e.g., [9, Theorem 7.3.1], there exists $Q \in O(n)$ and $S \in \mathbb{S}^n_+$ such that $[X\ 0] = SQ$. Now, partition $Q = [U\ W]$ according to $[X\ 0]$. Then

$$XU^T = [X\ 0] \cdot \begin{bmatrix} U^T \\ W^T \end{bmatrix} = S \succeq 0,$$

and, by construction, U has orthonormal columns. □

Polarizability can be expressed in terms of the subdifferential of the nuclear norm.

Lemma 3.3. *Let $X \in \mathbb{R}^{n\times p}$ and let $U \in \mathcal{V}_{n,p}$. The following are equivalent:*

(i) *$U \in \partial\|\cdot\|_*(X)$;*
(ii) *$\langle U, X\rangle = \|X\|_*$;*
(iii) *U polarizes X, i.e. $XU^T \in \mathbb{S}^n_+$.*

Proof. (i)⇒(ii): By the subdifferential representation of $\|\cdot\|_*$ in (7).
(ii)⇒(iii): Observe that $U^TU = I_p$. In particular, $\sigma(U) = [1,\dots,1]^T \in \mathbb{R}^p$. By assumption, we hence have $\langle U, X\rangle = \|X\|_* = \langle \sigma(U), \sigma(X)\rangle$. By (von Neumann's) Theorem 2.3, we thus find $\bar U \in O(n), \bar V \in O(p)$ such that $X = \bar U \mathrm{diag}(\sigma(X))\bar V^T$ and $U = \bar U \left(\begin{smallmatrix} I_p & 0 \\ 0 & 0\end{smallmatrix}\right) \bar V^T$. Consequently

$$XU^T = \bar U \begin{pmatrix} \mathrm{diag}(\sigma(\bar X)) & 0 \\ 0 & 0\end{pmatrix} \bar V^T \bar V \left(\begin{smallmatrix} I_p & 0 \\ 0 & 0\end{smallmatrix}\right) \bar U^T = \bar U \begin{pmatrix} \mathrm{diag}(\sigma(\bar X)) & 0 \\ 0 & 0\end{pmatrix} \bar U^T \succeq 0.$$

(iii)⇒(i): Extend U to a an orthonormal matrix $[U\ W] \in O(n)$. Then

$$\mathrm{tr}\,(XU^T) = \|XU^T\|_* = \|X\|_*,$$

where the first identity employs the assumption that $XU^T \in \mathbb{S}^n_+$ combined with (3) and the second one is due to Lemma 2.5. Since $\|U\|_{op} = 1$ (as $U^TU = I_p$), the desired statement follows from Proposition 2.6(b). □

We now present our first main result which characterizes the existence of (proper) line segments in the nuclear norm sphere.

Theorem 3.4 (Flats in the nuclear norm sphere)**.** *Let $\bar X, \hat X \in \mathbb{R}^{n\times p}$ and define*

$$X(t) := \bar X + t(\hat X - \bar X) \quad \forall t \in [0,1].$$

Then the following are equivalent:

(i) *$\|X(t)\|_* = \|X\|_*$ for all $t \in [0,1]$.*
(ii) *$\hat X$ and $\bar X$ are simultaneously polarizable and $\|\bar X\|_* = \|\hat X\|_*$.*

Proof. Note that there is nothing to prove if $\bar X = \hat X$. So we assume the contrary from now on.
(i)⇒(ii): By assumption, the convex function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \|X(t))\|_*$, is constant on $[0,1]$. Hence, by the (subdifferential) chain rule [16, Theorem 23.8], we have

$$\{0\} = \{f'(t)\} = \left\{\left\langle \hat X - \bar X, Y\right\rangle \mid Y \in \partial\|\cdot\|_*(X(t))\right\} \quad \forall t \in (0,1),$$

i.e.

$$\left\langle \hat X - \bar X, Y\right\rangle = 0 \quad \forall Y \in \partial\|\cdot\|_*(X(t)),\ t \in (0,1). \tag{11}$$

Now, for any $t \in (0,1)$ and any $Y \in \partial\|\cdot\|_*(X(t))$, we have

$$\left\langle t\hat{X} + (1-t)\bar{X},\, Y\right\rangle = \langle X(t),\, Y\rangle = \|X(t)\|_* = \|\bar{X}\|_*. \tag{12}$$

Multiplying (11) by $-t$ and adding to (12) then yields

$$\left\langle \bar{X},\, Y\right\rangle = \|\bar{X}\|_* \quad \forall Y \in \partial\|\cdot\|_*(X(t)),\ t \in (0,1),$$

hence

$$\partial\|\cdot\|_*(X(t)) \subset \partial\|\cdot\|_*(\bar{X}) \quad \forall t \in (0,1). \tag{13}$$

Similarly, multiplying (11) by $(1-t)$ and adding to (12) ultimately yields

$$\partial\|\cdot\|_*(X(t)) \subset \partial\|\cdot\|_*(\hat{X}) \quad \forall t \in (0,1). \tag{14}$$

Combining (13) and (14) we thus find

$$\partial\|\cdot\|_*(X(t)) \subset \partial\|\cdot\|_*(\hat{X}) \cap \partial\|\cdot\|_*(\bar{X}) \quad \forall t \in (0,1). \tag{15}$$

Now, for $t \in (0,1)$, set $X_t := X(t)$. Choose $U_t \in \mathbb{R}^{n\times p}$ that polarizes X_t by means of Proposition 3.2. Then, by Lemma 3.3 , we have $U_t \in \partial\|\cdot\|_*(X_t)$, and consequently, by (15), we find $U_t \in \partial\|\cdot\|_*(\hat{X}) \cap \partial\|\cdot\|_*(\bar{X})$. Therefore, we find

$$\left\langle \bar{X},\, U_t\right\rangle = \|\bar{X}\|_* \ \text{ and } \ \left\langle \hat{X},\, U_t\right\rangle = \|\hat{X}\|_*.$$

By Lemma 3.3 we thus infer that U_t polarizes both $\bar{X}$ and $\hat{X}$.

(ii) $\Rightarrow$ (i): Let $U \in \mathbb{R}^{n\times p}$ polarize $\bar{X}$ and $\hat{X}$. Consequently, U polarizes $X(t)$, and hence, by Lemma 3.3, $U \in \partial\|\cdot\|_*(X(t))$ for all $t \in [0,1]$. Therefore

$$\|X(t)\|_* = \operatorname{tr}(X(t)U^T) = t\cdot\operatorname{tr}(\bar{X}U^T) + (1-t)\cdot\operatorname{tr}(\hat{X}U^T) = \|\bar{X}\|_*.$$

Here, the last identity uses that $\operatorname{tr}(\bar{X}U^T) = \|\bar{X}\|_* = \|\hat{X}\|_* = \operatorname{tr}(\hat{X}U^T)$ as U polarizes both $\bar{X}$ and $\hat{X}$ which have the same nuclear norm (by assumption). □

An immediate consequence is the following corollary.

Corollary 3.5. *For $\bar{X}, \hat{X} \in \mathbb{R}^{n\times p}$ and $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n\times p}, \mathbb{E})$ the following are equivalent:*

(i) *$\|\cdot\|_*$ and $\mathcal{A}$ are constant on the line segment $[\bar{X}, \hat{X}]$.*
(ii) *$\hat{X} - \bar{X} \in \ker\mathcal{A}$, $\|\bar{X}\|_* = \|\hat{X}\|_*$ and $\hat{X}$ and $\bar{X}$ are simultaneously polarizable.*

The previous result, while geometrically elegant, is potentially difficult to evaluate. By working with the singular value decomposition of a the base point $\bar{X}$, one can further specify exactly the set of directions which should not be contained in the kernel of the ambient linear operator $\mathcal{A}$. To state this result, we use the following notation for some $r \in \{1,\dots,n\}$:

$$\mathcal{S}^r_{++} := \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{S}^n_+ \;\middle|\; A \in \mathbb{S}^r_{++} \right\} \subset \mathbb{S}^n_+.$$

Corollary 3.6. *Let $\bar{X} \in \mathbb{R}^{n\times p}$ and let $r := \operatorname{rank}\bar{X}$. Let there be posed a singular value decomposition $\bar{X} = \bar{U}\operatorname{diag}(\sigma(\bar{X}))\bar{V}^T$ and let $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n\times p}, \mathbb{E})$. Set*

$$W(\bar{X}) := \left\{ \bar{U}M\left(\begin{smallmatrix} I_r & 0 \\ 0 & R \end{smallmatrix}\right)\bar{V}^T \;\middle|\; \begin{array}{l} M \in \mathbb{S}^n_+ - \mathcal{S}^r_{++},\ \operatorname{tr}(M) = 0, \\ R \in \mathcal{V}_{n-r,p-r},\ M\begin{pmatrix} I_r & 0 \\ 0 & RR^T \end{pmatrix} = M \end{array} \right\}.$$

See Section 3.1 for a discussion of W. The following are equivalent:

(i) $\mathcal{X} := \left\{ X \in \mathbb{R}^{n\times n} \;\middle|\; \mathcal{A}(X) = \mathcal{A}(\bar X),\ \|X\|_* = \|\bar X\|_* \right\}$ *does not contain a proper*[6] *line segment including* $\bar X$*.*
(ii) $\ker \mathcal{A} \cap W(\bar X) = \{0\}$.

Proof. (ii)⇒(i): Assume (i) does not hold, i.e. there is $\hat X \neq \bar X$ such that $[\bar X, \hat X] \subset \mathcal{X}$. By Corollary 3.5, we find that $\bar X$ and $\hat X$ are simultaneously polarizable, i.e. there exists $U \in \mathcal{V}_{n,p}$ such that $\bar X U^T \in \mathbb{S}^n_+$ and $\hat X U^T \in \mathbb{S}^n_+$. In particular, by Lemma 3.3, $U \in \partial \|\cdot\|_*(\bar X)$, hence, by Proposition 2.8, $U = \bar U \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \bar V^T$ for some $R \in \mathcal{V}_{n-r,p-r}$. The latter comes from the fact that $U \in \mathcal{V}_{n,p}$. Moreover, since $\hat X U^T \in \mathbb{S}^n_+$, we find that

$$\bar U^T \hat X \bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix} = \bar U^T(\hat X U^T)\bar U \in \mathbb{S}^n_+. \tag{16}$$

On the other hand, we also have

$$\bar U^T \bar X \bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix} = \operatorname{diag}(\sigma(\bar X)) \in \mathcal{S}^r_{++}.$$

Combining this with (16), we find that

$$M := \bar U^T(\hat X - \bar X)\bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix} \in \mathbb{S}^n_+ - \mathcal{S}^r_{++},$$

and, trivially, $M \begin{pmatrix} I_r & 0 \\ 0 & RR^T\end{pmatrix} = M$. Moreover

$$\begin{aligned} \operatorname{tr}(M) &= \operatorname{tr}\left(\bar U^T \hat X \bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix}\right) - \operatorname{tr}\left(\bar U^T \bar X \bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix}\right) \\ &= \left\|\bar U^T \hat X \bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix}\right\|_* - \left\|\bar U^T \bar X \bar V \begin{pmatrix} I_r & 0 \\ 0 & R^T\end{pmatrix}\right\|_* \\ &= \|\hat X\|_* - \|\bar X\|_* \\ &= 0, \end{aligned}$$

where the second identity uses the positive semidefiniteness of the matrices in question (combined with (3)), and the third one uses orthogonal invariance and Lemma 2.5. Since also $\mathcal{A}(\hat X) = \mathcal{A}(\bar X)$, we consequently have

$$0 \neq \hat X - \bar X = \bar U M \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \bar V^T \in W(\bar X) \cap \ker \mathcal{A}.$$

(i)⇒(ii): Assume (ii) does not hold, i.e. there exists $M \in (\mathbb{S}^n_+ - \mathcal{S}^r_{++})\backslash\{0\}$ with $\operatorname{tr}(M) = 0$ and $R \in \mathcal{V}_{n-r,p-r}$ such that $Y := \bar U M \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \bar V^T \in \ker \mathcal{A}$. Define $U := \bar U \left(\begin{smallmatrix} I_r & 0 \\ 0 & R\end{smallmatrix}\right) \bar V^T$. Since $\|R\|_{op} = 1$, in view of Proposition 2.8, we have $U \in \partial\|\cdot\|_*(\bar X)$. Now, for $\varepsilon > 0$ set $X(\varepsilon) := \bar X + \varepsilon Y$. Then

$$\begin{aligned} \bar U^T(X(\varepsilon)U^T)\bar U &= \bar U^T \bar X U^T \bar U + \varepsilon \bar U^T Y U^T \bar U \\ &= \operatorname{diag}(\sigma(\bar X)) + \varepsilon M \begin{pmatrix} I_r & 0 \\ 0 & RR^T\end{pmatrix} \\ &= \operatorname{diag}(\sigma(\bar X)) + \varepsilon M. \end{aligned}$$

Recall that $M \in \mathbb{S}^n_+ - \mathcal{S}^r_{++}$, and $\operatorname{diag}(\sigma(\bar X)) \in \mathcal{S}^r_{++}$, and hence we can find $\hat\varepsilon > 0$, sufficiently small, such that $\operatorname{diag}(\sigma(\bar X)) + \hat\varepsilon M \in \mathbb{S}^n_+$. Consequently, for $\hat X := X(\hat\varepsilon)$, we have $\hat X U^T \in \mathbb{S}^n_+$, i.e. U polarizes $\hat X$ (and $\bar X$). In addition, we find that

$$\|\hat X\|_* = \operatorname{tr}(\bar U^T(\hat X U^T)\bar U) = \operatorname{tr}(\operatorname{diag}(\sigma(\bar X))) + \hat\varepsilon \cdot \operatorname{tr}(M) = \|\bar X\|_*,$$

[6] A line segment that is not just $\{\bar X\}$.

as $\mathrm{tr}\,(M) = 0$. Since we have $\mathcal{A}(\hat{X}) = \mathcal{A}(\bar{X})$ as well, Corollary 3.5 now gives the desired conclusion. □

3.1. **The set $W(\bar{X})$ in Corollary 3.6.** Some comments on the contents of Corollary 3.6 are in order. We note that that the set $W(\bar{X})$ is a cone, owing to the set $\mathbb{S}^n_+ - \mathcal{S}^r_{++}$ being a cone. While the cone is not reflection symmetric, the condition (ii) is equivalently formulated with the symmetrization of $W(\bar{X})$ under the reflection $x \mapsto -x$. The symmetrization of $W(\bar{X})$ has the interpretation as the subset of the tangent space at $\bar{X}$ (in $\mathbb{R}^{n\times p}$) in which the nuclear norm changes linearly (i.e. is non-strictly convex). The cone $W(\bar{X})$ is *not* generally convex, save for the case that $\bar{X}$ is full rank; in that case, the set $W(\bar{X})$ simplifies to

$$W(\bar{X}) := \left\{\bar{U}\ M[I_p\ 0]^T\ \bar{V}^T \;\middle|\; M \in \mathbb{S}^n,\ \mathrm{tr}\,(M) = 0\right\}.$$

Moreover, we point out that in the square case ($n = p$), the set $W(\bar{X})$ simplifies to

$$W(\bar{X}) = \left\{\bar{U}M \left(\begin{smallmatrix} I_r & 0 \\ 0 & R \end{smallmatrix}\right) \bar{V}^T \;\middle|\; M \in \mathbb{S}^n_+ - \mathcal{S}^r_{++},\ \mathrm{tr}\,(M) = 0,\ R \in O(n-r)\right\}.$$

While $W(\bar{X})$ is relatively pathological, we note that its span has a simple expression:

Proposition 3.7 ($\mathrm{span}\, W(\bar{X})$)**.** *Let $\bar{X} \in \mathbb{R}^{n\times p}$ and let $r := \mathrm{rank}\,\bar{X}$. Let there be posed a singular value decomposition $\bar{X} = \bar{U}\mathrm{diag}(\sigma(\bar{X}))\bar{V}^T$ and let $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n\times p}, \mathbb{E})$. Let $W(\bar{X})$ be as in Corollary 3.6. Then*

$$\mathrm{span}\, W(\bar{X}) = \left\{\bar{U} \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \bar{V}^T \;\middle|\; A \in \mathbb{S}^r, B \in \mathbb{R}^{r\times(p-r)}, C \in \mathbb{R}^{(n-r)\times r}, D \in \mathbb{R}^{(n-r)\times(p-r)}\right\}.$$

Proof. Let W_4 be the right-hand side of the displayed equation. The containment of $\mathrm{span}\, W(\bar{X}) \subset W_4$ is immediate from the containment $W(\bar{X}) \subset W_4$ and the fact that the latter is a subspace. For the reverse, we argue by construction of a flag $W_1 \subset W_2 \subset W_3 \subset W_4$ each of which we show is in $\mathrm{span}\, W(\bar{X})$. Set

$$\begin{aligned} W_1 &:= \left\{\bar{U} \left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right) \bar{V}^T \;\middle|\; D \in \mathbb{R}^{(n-r)\times(p-r)}\right\}, \\ W_2 &:= \left\{\bar{U} \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \bar{V}^T \;\middle|\; A \in \mathbb{S}^r, D \in \mathbb{R}^{(n-r)\times(p-r)}\right\}, \\ W_3 &:= \left\{\bar{U} \left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right) \bar{V}^T \;\middle|\; A \in \mathbb{S}^r, C \in \mathbb{R}^{(n-r)\times r}, D \in \mathbb{R}^{(n-r)\times(p-r)}\right\}. \end{aligned} \tag{17}$$

<u>$W_1 \subset \mathrm{span}\, W(\bar{X})$</u>: For any element of W_1 for some $D \in \mathbb{R}^{(n-r)\times(p-r)}$, let $R \in \mathcal{V}_{n-r,p-r}$ be a polarizing matrix such that $P = DR^T \in \mathbb{S}^{n-r}_+$. Now set

$$M := \left(\begin{smallmatrix} aI_r & 0 \\ 0 & P \end{smallmatrix}\right) \quad \text{where} \quad a := -\frac{\mathrm{tr}\,(P)}{r}.$$

Then

$$\tfrac{1}{2}M \left(\begin{smallmatrix} I_r & 0 \\ 0 & R \end{smallmatrix}\right) - \tfrac{1}{2}M \left(\begin{smallmatrix} I_r & 0 \\ 0 & -R \end{smallmatrix}\right) = M \left(\begin{smallmatrix} 0 & 0 \\ 0 & R \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right). \tag{18}$$

Since $PRR^T = DR^TRR^T = DR^T = P$ and $\mathrm{tr}\,(M) = 0$, we conclude that $W_1 \subset \mathrm{span}\, W(\bar{X})$.

<u>$W_2 \subset \mathrm{span}\, W(\bar{X})$</u>: It suffices to show that $W_2/W_1 \subset \mathrm{span}\, W(\bar{X})/W_1$. For arbitrary $A \in \mathbb{S}^r$, let $P_1, P_2 \in \mathbb{S}^r_{++}$ be such that $A = P_1 - P_2$. Now set

$$M_i := \begin{pmatrix} -P_i & 0 \\ 0 & a_i I_{n-r} \end{pmatrix} \quad \text{where} \quad a_i = \tfrac{\mathrm{tr}\,(P_i)}{n-r} \quad i = 1, 2.$$

Then $\mathrm{tr}\,(M_i) = 0$ ($i = 1, 2$), and taking $R = [I_{p-r}\ 0]^T$, we have that

$$\left(\begin{smallmatrix} A & 0 \\ 0 & * \end{smallmatrix}\right) = -M_1 \left(\begin{smallmatrix} I_r & 0 \\ 0 & R \end{smallmatrix}\right) + M_2 \left(\begin{smallmatrix} I_r & 0 \\ 0 & R \end{smallmatrix}\right) \in \mathrm{span}\, W(\bar{X}),$$

where the $(*)$ represents a matrix of no importance. Therefore, $W_2/W_1 \subset \operatorname{span} W(\bar{X})/W_1$, which suffices to show the desired inclusion.

$W_3 \subset \operatorname{span} W(\bar{X})$: It suffices to show that $W_3/W_2 \subset \operatorname{span} W(\bar{X})/W_2$. Take an arbitary element W_3/W_2 represented by some matrix $C \in \mathbb{R}^{(n-r)\times r}$. For some c sufficiently large

$$A = \begin{pmatrix} -\frac{n-r}{r} cI_r & C^T \\ C & cI_{n-r} \end{pmatrix} \in \{M \in \mathbb{S}^n_+ - \mathcal{S}^r_{++} \mid \operatorname{tr}(M) = 0\}.$$

Taking $R = [I_{p-r}\ 0]^T$, we find

$$\tfrac{1}{2} A \begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix} + \tfrac{1}{2} A \begin{pmatrix} I_r & 0 \\ 0 & -R \end{pmatrix} = A \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \in W(\bar{X}),$$

and hence $W_3/W_2 \in \operatorname{span} W(\bar{X})/W_2$.

$W_4 \subset \operatorname{span} W(\bar{X})$: It suffices to show that $W_4/W_3 \subset \operatorname{span} W(\bar{X})/W_3$. Take an arbitary element W_4/W_3 represented by some matrix $B \in \mathbb{R}^{r\times(p-r)}$. Then for all c sufficiently large,

$$A = \begin{pmatrix} -\frac{p-r}{r} cI_r & B & 0 \\ B^T & cI_{p-r} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \{M \in \mathbb{S}^n_+ - \mathcal{S}^r_{++} \mid \operatorname{tr}(M) = 0\}.$$

Taking $R = [I_{p-r}\ 0]^T$, we find

$$A \begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} -\frac{p-r}{r} cI_r & B \\ B^T & cI_{p-r} \\ 0 & 0 \end{pmatrix} \in W(\bar{X}),$$

and hence we conclude that $W_4/W_3 \in \operatorname{span} W(\bar{X})/W_3$.

This concludes the proof. □

4. Unique solutions in nuclear norm minimization

Throughout this section, let $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ be a (finite-dimensional) Euclidean space and let $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n\times p}, \mathbb{E})$. Our study above immediately yields uniqueness results for the nuclear norm minimization

$$\min_{X\in\mathbb{R}^{n\times p}} \|X\|_* \text{ s.t. } \mathcal{A}(X) = b. \tag{19}$$

Corollary 4.1. *Let $\bar{X} \in \mathbb{R}^{n\times p}$ be a solution of* (19) *with* $\operatorname{rank} \bar{X} = r$, *and let $W(\bar{X})$ be defined as in Corollary 3.6. Then the following are equivalent:*

(i) *$\bar{X}$ is the unique solution of* (19).

(ii) $\ker \mathcal{A} \cap W(\bar{X}) = \{0\}$.

Proof. Observe that the solution set $\mathcal{X}$ of (19) is convex and can be written as $\mathcal{X} = \{X \in \mathbb{R}^{n\times n} \mid \mathcal{A}(X) = \mathcal{A}(\bar{X}),\ \|X\|_* = \|\bar{X}\|_*\}$. Now, by convexity, $\mathcal{X}$ does not contain a proper line segment including $\bar{X}$ if and only if $\bar{X}$ is the unique solution of (19). Thus Corollary 3.6 gives the desired statement. □

4.1. **Sufficient conditions through convex analysis.** The following result is a generic convex analysis result which, given a solution, provides a sufficient condition for uniqueness of solutions to a(ny) convex optimization problem. It was established in [6] that in the polyhedral convex case it is also necessary which was then exploited to establish uniqueness of solutions for ℓ_1-minimization problems.

Proposition 4.2. *Let $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and assume that $0 \in \operatorname{int} \partial f(\bar{x})$. Then* $\operatorname{argmin} f = \{\bar{x}\}$.

Proof. Let $x \in \mathbb{E}$. By assumption, there exists $\varepsilon > 0$ such that $\varepsilon(x - \bar{x}) \in \partial f(\bar{x})$. Consequently

$$f(x) \geqslant f(\bar{x}) + \langle \varepsilon(x - \bar{x}),\, x - \bar{x}\rangle = f(\bar{x}) + \varepsilon\|x - \bar{x}\|^2 > f(\bar{x}).$$

□

We will, of course, apply this to the objective function $f = \|\cdot\|_* + \delta_{\{0\}}(\mathcal{A}(\cdot) - b)$ of (19). It turns out that the following conditions at some (feasible) point $\bar{X}$ are equivalent to having $0 \in \operatorname{int}(\partial f(\bar{X}))$.

Assumption 4.3. *For $\bar{X} \in \mathbb{R}^{n\times p}$ such that $\mathcal{A}(\bar{X}) = b$ it holds that:*
(i) $\operatorname{ri}(\partial\|\cdot\|_*(\bar{X})) \cap \operatorname{rge}\mathcal{A}^* \neq \emptyset$;
(ii) $\operatorname{par}\big(\partial\|\cdot\|_*(\bar{X})\big) + \operatorname{rge}\mathcal{A}^* = \mathbb{R}^{n\times p}$.

The reader can make these conditions even more tangible by inserting the respective expressions for the relative interior and parallel subspace of the $\partial\|\cdot\|_*(\bar{X})$ provided in Proposition 2.8 (and (10)).

We now provide the advertized characterization.

Proposition 4.4. *Let $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n\times p}, \mathbb{E})$, $b \in \mathbb{E}$ and define the (closed) proper, convex function $f : \mathbb{R}^{n\times p} \to \mathbb{R} \cup \{+\infty\}$ by $f(X) = \|X\|_* + \delta_{\{0\}}(\mathcal{A}(X) - b)$. For $\bar{X} \in \mathbb{R}^{n\times p}$ such that $\mathcal{A}(\bar{X}) = b$, the following are equivalent:*
(I) $0 \in \operatorname{int}\partial f(\bar{X})$.
(II) *Assumption 4.3 holds at $\bar{X}$.*

Proof. Observe that $\partial(\delta_{\{0\}}((\cdot) - b) \circ \mathcal{A})(\bar{X}) = \mathcal{A}^*\partial\delta_{\{0\}}(0) = \mathcal{A}^*\mathbb{E} = \operatorname{rge}\mathcal{A}^*$, by the chain rule [16, Theorem 23.9], and consequently $\partial f(\bar{X}) = \partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*$, by the sum rule [16, Theorem 23.8]. Hence (I) reads

$$0 \in \operatorname{int}(\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*) = \operatorname{ri}(\partial\|\cdot\|_*(\bar{X})) + \operatorname{rge}\mathcal{A}^*,$$

where the identity uses the sum rule for the relative interior [16, Corollary 6.6.2] and the fact that a subspace is relatively open. This already shows that (I) implies $\operatorname{ri}(\partial\|\cdot\|_*(\bar{X})) \cap \operatorname{rge}\mathcal{A}^* \neq \emptyset$. On the other hand, it also yields that, for any $y \in \partial\|\cdot\|_*(\bar{X})$, we have

$$\begin{aligned}\mathbb{R}^{n\times p} &= \operatorname{aff}(\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*)\\ &= \operatorname{aff}(\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*\\ &= \operatorname{aff}(\partial\|\cdot\|_*(\bar{X}) - y) + y + \operatorname{rge}\mathcal{A}^*\\ &= \operatorname{par}\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*.\end{aligned}$$

All in all, (I) implies (II).

Conversely, if (II), starting from $\operatorname{par}\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*$, the latter equations shows $\mathbb{R}^{n\times p} = \operatorname{aff}(\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*)$, while $\operatorname{ri}(\partial\|\cdot\|_*(\bar{X})) \cap \operatorname{rge}\mathcal{A}^* \neq \emptyset$ implies

$$0 \in \operatorname{ri}(\partial\|\cdot\|_*(\bar{X})) + \operatorname{rge}\mathcal{A}^* = \operatorname{ri}(\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*) = \operatorname{int}(\partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^*),$$

where the first identity is, again, due to the sum rule for the relative interior, while the last identity uses the fact that the relative interior is an interior if (and only if) the parallel subspace (which is here equal to the affine hull) of the convex set in question is the whole space. □

Corollary 4.5. *Let $\bar{X}$ be a solution of* (19) *such that Assumption 4.3 holds at $\bar{X}$. Then $\bar{X}$ is the unique solution of* (19)

Proof. Combine Proposition 4.2 and Proposition 4.4. □

4.2. **More insight.** Combining Corollary 4.5 and Corollary 3.6, it follows readily that Assumption 4.3 at $\bar{X}$ implies that $W(\bar{X}) \cap \ker \mathcal{A} = \{0\}$. On the other hand, this argument is not very illuminating when trying to understand the exact interplay of these two types of conditions. Moreover, it is not clear whether Assumption 4.3 might also be necessary for uniqueness of solutions (as it is for its ℓ_1-analog). We shed some light on these issues now and start with an auxiliary result.

Lemma 4.6. *Let $\bar{X}$ satisfy Assumption 4.3. Then*

$$\mathbb{R}^{n\times p} = \operatorname{rge} \mathcal{A}^* + \mathbb{R}_+ \partial \|\cdot\|_*(\bar{X}).$$

Proof. Set $\mathcal{S} := \partial\|\cdot\|_*(\bar{X})$, and let $Y \in \operatorname{rge}\mathcal{A}^* \cap \operatorname{ri}\mathcal{S}$ which exists by Assumption 4.3 (i). Then

$$\begin{aligned} \mathbb{R}^{n\times p} &= \operatorname{rge}\mathcal{A}^* + \operatorname{par}\left(\partial\|\cdot\|_*(\bar{X})\right) \\ &= \operatorname{rge}\mathcal{A}^* + \mathbb{R}_+(\mathcal{S} - Y) \\ &= \mathbb{R}_+\left(\operatorname{rge}\mathcal{A}^* + \mathcal{S} - Y\right) \\ &= \mathbb{R}_+(\operatorname{rge}\mathcal{A}^* + \mathcal{S}) \\ &= \operatorname{rge}\mathcal{A}^* + \mathbb{R}_+\mathcal{S}. \end{aligned}$$

Here the second identity uses the property of relative interior points from (9). □

As alluded to above, the following result is clear from our previous analysis. We give an explicit proof in the hopes of consolidating the different flavors of the conditions in Assumption 4.3 and Corollary 3.6, respectively.

Proposition 4.7. *Let $\bar{X} \in \mathbb{R}^{n\times p}$ with $r := \operatorname{rank}\bar{X}$ and singular value decomposition $\bar{X} = \bar{U}\operatorname{diag}(\sigma(\bar{X}))\bar{V}^T$. If Assumption 4.3 holds, then* $\ker\mathcal{A} \cap W(\bar{X}) = \{0\}$.

Proof. Let $X \in \ker\mathcal{A} \cap W(\bar{X})$. Then, by definition of $W(\bar{X})$, there exists $M = \begin{pmatrix} A-D & B \\ B^T & C \end{pmatrix}$ with $A \in \mathbb{S}^r_+, C \in \mathbb{S}^{n-r}_+$, $D \in \mathbb{S}^r_{++}$, $\operatorname{tr}(A) + \operatorname{tr}(C) = \operatorname{tr}(D)$, and $R \in \mathcal{V}_{n-r,p-r}$ such that

$$X = \bar{U}M\begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix}\bar{V}^T.$$

On the other hand, by Lemma 4.6 and Proposition 2.8 we find $Z \in \operatorname{rge}\mathcal{A}^*$, $F \in \mathbb{B}_{op}$ and $t \geqslant 0$ such that

$$X = Z + t \cdot \bar{U}\begin{pmatrix} I_r & 0 \\ 0 & F \end{pmatrix}\bar{V}^T.$$

Consequently, we have

$$\begin{aligned}
\|X\|^2 &= \left\langle \bar{U}M \begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix} \bar{V}^T,\, Z + t\cdot \bar{U}\begin{pmatrix} I_r & 0 \\ 0 & F \end{pmatrix}\bar{V}^T \right\rangle \\
&= t\cdot \left\langle \bar{U}M \begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix} \bar{V}^T,\, \bar{U}\begin{pmatrix} I_r & 0 \\ 0 & F \end{pmatrix}\bar{V}^T \right\rangle \\
&= t\cdot \left\langle M \begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix},\, \begin{pmatrix} I_r & 0 \\ 0 & F \end{pmatrix} \right\rangle \\
&= t\cdot \operatorname{tr}\left(\begin{pmatrix} I_r & 0 \\ 0 & RF^T \end{pmatrix} \cdot \begin{pmatrix} A-D & B \\ B^T & C \end{pmatrix} \right) \\
&= t\cdot \operatorname{tr}\left(\begin{pmatrix} A-D & B \\ RF^TB^T & RF^TC \end{pmatrix} \right) \\
&= t\cdot \left(\operatorname{tr}(A) - \operatorname{tr}(D) + \operatorname{tr}(RF^TC) \right) \\
&= t\cdot \left(\operatorname{tr}(RF^TC) - \operatorname{tr}(C) \right) \\
&\leqslant t\cdot \left(\|RF^T\|_{op}\cdot \|C\|_* - \|C\|_* \right) \\
&\leqslant 0.
\end{aligned}$$

Here, the second identity takes into account that $Z \in \operatorname{rge}\mathcal{A}^*$ while $\bar{U}M\begin{pmatrix} I_r & 0 \\ 0 & R \end{pmatrix}\bar{V}^T = X \in \ker\mathcal{A}$. The seventh (last) equality uses the fact that $\operatorname{tr}(A) + \operatorname{tr}(C) = \operatorname{tr}(D)$. The first inequality uses the fact that C is positive semidefinite as well as the 'Hölder inequality' for the operator and nuclear norm. The last inequality is due to the fact that $\|R\|_{op} = 1$, $\|F\|_{op} \leqslant 1$ and the submulitiplicativity of the operator norm.

All in all, we find that $X = 0$ which proves the desired result. □

The natural question as to whether Assumption 4.3 is also necessary for uniqueness is answered negatively by the following example.

Example 4.8. *Set* $\mathbb{E} := \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$, *and define* $\mathcal{A} : \mathbb{R}^{2\times 2} \to \mathbb{E}$ *by*

$$\mathcal{A}(X) = \left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} X,\ P_{\mathbb{A}^2}(X)\right],$$

where $P_{\mathbb{A}^2}(X) := \frac{1}{2}(X - X^T)$ *is the projection onto the* 2×2 *skew symmetric matrices* $\mathbb{A}^2$. *Set* $b := \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] \in \mathbb{E}$. *Equipped with these choices, consider*

$$\min_{X\in\mathbb{R}^{2\times 2}} \|X\|_* \ \text{s.t.}\ \mathcal{A}(X) = b. \tag{20}$$

The following hold:

- $\ker\mathcal{A} = \operatorname{span}\left\{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right\}$.
- $\mathcal{A}^* : \mathbb{E} \to \mathbb{R}^{2\times 2}, \quad \mathcal{A}^*(Y, Z) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} Y + P_{\mathbb{A}^2}(Z)$.
- $\operatorname{rge}\mathcal{A}^* = \left\{\begin{pmatrix} t & s \\ t & s \end{pmatrix} \mid t, s \in \mathbb{R}\right\} + \mathbb{A}^2$.

Now, set $\bar{X} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. *Then* $\mathcal{A}(\bar{X}) = b$, *i.e.* $\bar{X}$ *is feasible for* (20). *Moreover, by Proposition 2.8, observe that*

$$\partial\|\cdot\|_*(\bar{X}) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \mid \beta \in [-1, 1]\right\} \quad \textit{and} \quad \operatorname{par}(\partial\|\cdot\|_*(\bar{X})) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \mid \beta \in \mathbb{R}\right\}.$$

It is then an easy exercise to find that $\operatorname{par}(\partial\|\cdot\|_*(\bar{X})) + \operatorname{rge}\mathcal{A}^* = \mathbb{R}^{2\times 2}$. *Moreover, we observe that*

$$\begin{aligned}
0 \in \partial\|\cdot\|_*(\bar{X}) + \operatorname{rge}\mathcal{A}^* &\iff \partial\|\cdot\|_*(\bar{X}) \cap \operatorname{rge}\mathcal{A}^* \neq \emptyset \\
&\iff \exists \beta \in [-1, 1], t, s, q \in \mathbb{R} :\ \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} t & s \\ t & s \end{pmatrix} + \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}.
\end{aligned}$$

The latter system has only one solution $t = q = 1, s = \beta = -1$. *In particular, we see that* $\bar{X}$ *is a minimizer of* (20) *and that*

$$\operatorname{ri}(\partial\|\cdot\|_*(\bar{X})) \cap \operatorname{rge}\mathcal{A}^* = \emptyset.$$

In particular, the sufficient condition from Assumption 4.3 for uniqueness is violated at $\bar{X}$ *(while (ii) is satisfied). In turn, realizing that* $\operatorname{rank}\bar{X} = 1$*, and consequently*

$$W(\bar{X}) = \left\{ \left(\begin{smallmatrix} a-d & b \\ b & c \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right) \,\middle|\, R^2 = 1, \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \mathbb{S}^2_+, \operatorname{tr}\left(\begin{smallmatrix} a-d & b \\ b & c \end{smallmatrix}\right) = 0, d > 0 \right\},$$

we find that

$$\begin{aligned} X \in W(\bar{X}) \cap \ker \mathcal{A} \quad &\Longrightarrow \quad X = \left(\begin{smallmatrix} x & -x \\ -x & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} a-d & \pm b \\ b & \pm c \end{smallmatrix}\right), a - d + c = 0. \\ &\Longrightarrow \quad X = 0. \end{aligned}$$

Therefore, by Corollary 3.6, $\bar{X}$ *is the unique solution of* (20).

4.3. **Other nuclear norm minimization problems.** The following general result affords us to carry over uniqueness results from above to other nuclear norm minimization problems involving a linear operator. The proof relies on *Fenchel-Rockafellar duality* [1, 8, 16, 17], and the dual correspondence of *strict convexity* and *essential smoothness* [16].

Proposition 4.9. *Let* $\mathcal{A} \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_2)$*,* $g : \mathbb{E} \to \mathbb{R}$ *strictly convex, and* $h : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ *closed, proper convex. Then* $\mathcal{A}$ *and* h *are constant on the solution set* $\mathcal{X}^* := \operatorname{argmin}_{x \in \mathbb{E}_1} \{g(\mathcal{A}(x)) + h(x)\}$.

Proof. Clearly, it suffices to prove that $\mathcal{A}$ is constant on $\mathcal{X}^*$. To this end, observe that the dual problem of (the primal problem) $\min_{x \in \mathbb{E}_1} \{g(\mathcal{A}(x)) + h(x)\}$ reads $\max_{y \in \mathbb{E}_2} \{-g^*(-y) - h^*(\mathcal{A}^*(y))\}$. Since g is finite-valued, strong duality holds, and, in particular, for some dual solution $\bar{y} \in \mathbb{E}_2$ and any primal solution $x \in \mathcal{X}^*$ it holds that, in particular, $\mathcal{A}(x) \in \partial g^*(-\bar{y})$, cf., e.g. [17, Example 11.41]. However, since g is strictly convex, g^* is *essentially smooth* [16, Theorem 26.3] and hence $\mathcal{A}(x) = \nabla g^*(-\bar{y})$. Since $x \in \mathcal{X}^*$ was arbitrary, this proves result. □

Corollary 4.10. *Let* $\mathcal{A} \in \mathcal{L}(\mathbb{R}^{n \times p}, \mathbb{E})$*,* $b \in \mathbb{E}$*,* $\lambda > 0$*,* $f : \mathbb{E} \to \mathbb{R}$ *strictly convex, and let* $\bar{X}$ *be a solution of*

$$\min_{X \in \mathbb{R}^{n \times n}} f(\mathcal{A}(X) - b) + \lambda \|X\|_*. \tag{21}$$

Then $\bar{X}$ *is the unique solution if and only if*

$$\left\{ X \in \mathbb{R}^{n \times p} \,\middle|\, \mathcal{A}(X) = \mathcal{A}(\bar{X}),\ \|X\|_* = \|\bar{X}\|_* \right\} = \{\bar{X}\}.$$

(all of which is the case if and only if $W(\bar{X}) \cap \ker \mathcal{A} = \{0\}$*).*

Proof. Let $\mathcal{X} = \operatorname{argmin}_{\mathbb{R}^{n \times p}} \{f(\mathcal{A}(\cdot) - b) + \lambda \| \cdot \|_*\}$ be the solution set of (21). Applying Proposition 4.9 to $g := f((\cdot) - b)$ and $h := \lambda \| \cdot \|_*$ yields that, in fact, $\mathcal{X} = \left\{ X \in \mathbb{R}^{n \times p} \,\middle|\, \mathcal{A}(X) = \mathcal{A}(\bar{X}),\ \|X\|_* = \|\bar{X}\|_* \right\}$. Therefore, the claim follows. □

5. Final remarks

In this paper, starting from a study of line segments in the nuclear norm sphere, we established necessary and sufficient conditions for uniqueness of solutions for minimizing the nuclear norm over an affine manifold. The central linear-algebraic notion in this regard is *simultaneous polarizability*, which formalizes the idea of rotating two (square) matrices in the same fashion to render them positive semidefinite. We then gave another set of sufficient conditions based on the convex geometry of the subdifferential (of the nuclear norm) and its interplay with (the range of) the ambient linear operator. A duality-based argument enabled us to transfer these findings to a whole class of nuclear norm-regularized optimization problems with strictly convex fidelity term.

As a topic of future research, we intend to build on this analysis to study stability of nuclear norm(-regularized) optimization problems in terms of the right-hand side b and the regularization parameter λ. In particular, we would like to study Lipschitz properties of the solution function

$$(b,\lambda) \mapsto \underset{X\in\mathbb{R}^{n\times p}}{\operatorname{argmin}} \left\{ \frac{1}{2}\|\mathcal{A}(X)-b\|^2 + \lambda\|X\|_* \right\}.$$

This study will rely on a suitable representation of the graph of the subdifferential of the nuclear norm.

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