Topics in Convex Analysis

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Spring School on Variational Analysis

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Outline

1. Fundamentals from Convex Analysis

The Euclidean setting and Minkowski notation

In what follows $\mathbb{E}$ will be a **Euclidean space**, i.e. a *real-vector space* equipped with an *inner product* $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ of dimension $\kappa < \infty$.

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- $\mathbb{E} = \mathbb{R}^n, \quad \langle x, y \rangle := x^T y, \quad \kappa = n$
- $\mathbb{E} = \mathbb{R}^{m \times n}, \quad \langle A, B \rangle := \mathrm{tr}\,(A^T B), \quad \kappa = mn$

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- $\mathbb{E}=\mathbb{R}^n, \quad \langle x,y\rangle := x^Ty, \quad \kappa=n$
- $\mathbb{E}=\mathbb{R}^{m\times n}, \quad \langle A,B\rangle := \mathrm{tr}\,(A^TB), \quad \kappa=mn$

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- $\mathrm{pos}\ S := \mathbb{R}_+ S \quad$ (conical hull)

Convex sets and cones

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$S \subset \mathbb{E}$ is said to be

- *convex* if $\lambda S + (1-\lambda)S \subset S \quad (\lambda \in (0,1))$;
- a *cone* if $\lambda S \subset S \quad (\lambda \geq 0)$.

Note that $K \subset \mathbb{E}$ is a *convex cone* iff $K + K \subset K$.

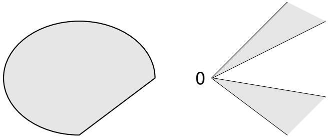

Figure: Convex set/non-convex cone

The convex hull and the closed convex hull

Definition 1 (Convex hull/closed convex hull).

Let $S \subset \mathbb{E}$ nonempty. Then the *convex hull* of S is the smallest convex set containing S, i.e.

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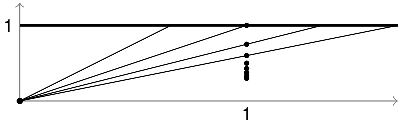
Example: $S := \{\binom{0}{0}\} \cup \{\binom{a}{1} \mid a \geq 0\}$,

$\binom{1}{1/k} = \frac{1}{k}\binom{k}{1} + \left(1 - \frac{1}{k}\right)\binom{0}{0} \in \operatorname{conv} S.$

But: $\binom{1}{1/k} \to \binom{1}{0} \notin \operatorname{conv} S.$



The topology relative to the affine hull

Affine set: A set $S = U + x$ with $x \in \mathbb{E}$ and a subspace $U \subset$ is called *affine*. This is characterized by

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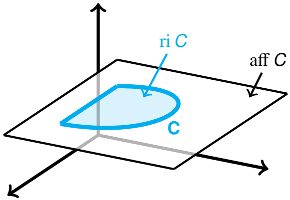

C	$\operatorname{aff} C$	$\operatorname{ri} C$
$\{x\}$	$\{x\}$	$\{x\}$
$[x, x']$	$\{\lambda x + (1-\lambda)x' \mid \lambda \in \mathbb{R}\}$	(x, x')
$\overline{B}_\varepsilon(x)$	$\mathbb{E}$	$B_\varepsilon(x)$

Table: Examples for relative interiors

The horizon cone

Definition 2 (Horizon cone).

For a nonempty set $S \subset \mathbb{E}$ the set

$$S^{\infty} := \{v \in \mathbb{E} \mid \exists \{x_k \in S\}, \{t_k\} \downarrow 0 : \; t_k x_k \to v\}$$

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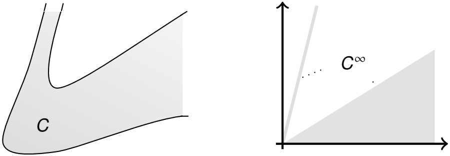

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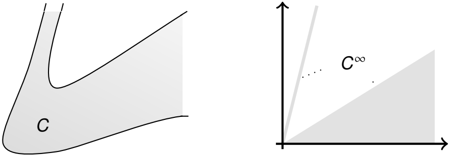

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Proposition 3 (The convex case).

Let $C \subset \mathbb{E}$ be nonempty and convex. Then $C^\infty = \{v \mid \forall x \in \operatorname{cl} C, \lambda \geq 0 : \ x + \lambda v \in \operatorname{cl} C\}$. In particular, C^∞ is (a closed and) convex (cone) if C is convex.

Extended real-valued functions: An epigraphical perspective

Let $f : \mathbb{E} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

- $\mathrm{epi}\, f := \left\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \;\middle|\; f(x) \leq \alpha\right\}$ (epigraph)

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$\rightarrow$ f is uniquely determined through $\operatorname{epi} f$!

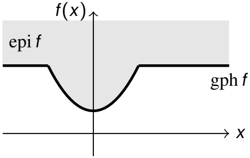

Figure: Epigraph of $f : \mathbb{R} \to \mathbb{R}$

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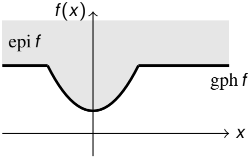

Figure: Epigraph of $f : \mathbb{R} \to \mathbb{R}$

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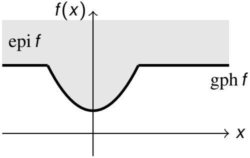

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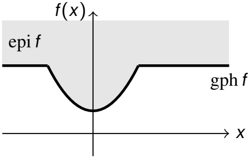

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Let $f : \mathbb{E} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

- $\mathrm{epi}\, f := \left\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \;\middle|\; f(x) \le \alpha\right\}$ (epigraph)
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$\rightarrow$ f is uniquely determined through $\mathrm{epi}\, f$!

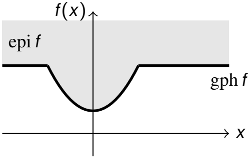

Figure: Epigraph of $f : \mathbb{R} \to \mathbb{R}$

f proper	$:\Leftrightarrow$	$-\infty < f \not\equiv +\infty$	$\Leftrightarrow$ [1]	$\mathrm{dom}\, f \neq \emptyset$
f convex	$:\Leftrightarrow$	$\mathrm{epi}\, f / \mathrm{epi}_{<} f$ convex	$\Leftrightarrow$ [1]	$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ $(x, y \in \mathbb{E},\ \lambda \in [0,1])$
f pos. hom.	$:\Leftrightarrow$ [1]	$\mathrm{epi}\, f$ cone	$\Leftrightarrow$	$\alpha f(x) = f(\alpha x)$ $(x \in \mathbb{E},\ \alpha \ge 0)$
f sublinear	$:\Leftrightarrow$ [1]	$\mathrm{epi}\, f$ cvx. cone	$\Leftrightarrow$	$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$ $(x, y \in \mathbb{E},\ \lambda, \mu \ge 0)$.

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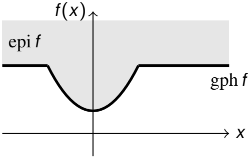

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			$\Leftrightarrow$	f convex $+$ positively homogeneous

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Lower semicontinuity

Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ and $\bar{x} \in \mathbb{E}$.

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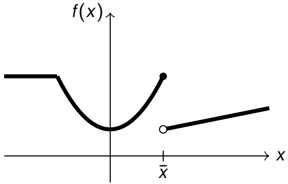

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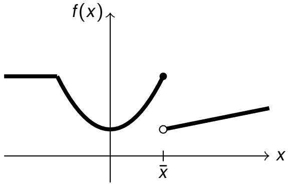

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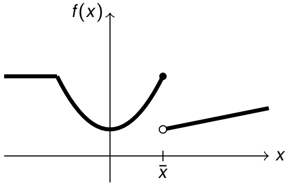

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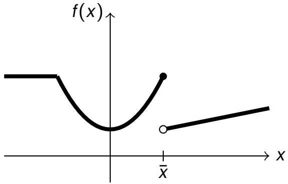

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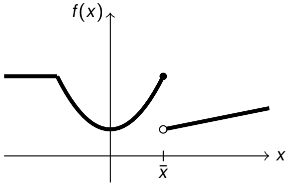

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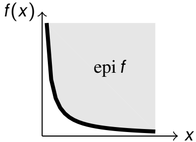

Figure: $f : x \mapsto \begin{cases} \frac{1}{x}, & x > 0, \\ +\infty, & \text{else.} \end{cases}$

Convexity preserving operations - new from old

1. Set Operations

For $C, C_i \ (i \in I) \subset \mathbb{E}, \ D \subset \mathbb{E}'$ convex, $F : \mathbb{E} \to \mathbb{E}'$ affine the following sets are convex:

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   - (Moreau envelope) $f:=e_\lambda g: x\mapsto \inf_u\left\{g(u)+\frac{1}{2\lambda}\|x-u\|^2\right\}$: $\mathrm{epi}\, f=\mathrm{epi}\, g+\mathrm{epi}\,\frac{1}{2}\|\cdot\|^2$.

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Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. A vector $v \in \mathbb{E}$ is called a *subgradient* of v at $\bar{x}$ if

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We denote by $\partial f(\bar{x})$ the set of all subgradients of f at $\bar{x}$ and call it the *(convex) subdifferential* of f at $\bar{x}$.

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- $\mathrm{ri}\,(\mathrm{dom}\, f)\subset\mathrm{dom}\,\partial f\subset\mathrm{dom}\, f$ (f convex).

Examples of subdifferentiation

- (Indicator function/Normal cone) Let $S \subset \mathbb{E}$.

Indicator function of S:

$$\delta_S : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}, \quad \delta_S(x) := \begin{cases} 0, & x \in S, \\ +\infty, & \text{else.} \end{cases}$$

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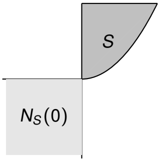

Figure: Normal cone

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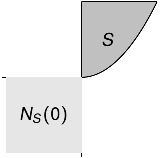

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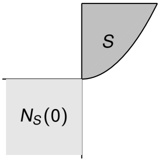

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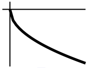
- (Empty subdifferential)

$$f : x \in \mathbb{R} \mapsto \begin{cases} -\sqrt{x} & \text{if} \quad x \geq 0, \\ +\infty & \text{else.} \end{cases}$$

$$\partial f(x) = \begin{cases} \left\{ -\frac{1}{2\sqrt{x}} \right\}, & x > 0, \\ \emptyset, & \text{else.} \end{cases}$$



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Interplay of conjugation and subdifferentiation

Theorem 6 (Subdifferential and conjugate function).

Let $f \in \Gamma_0$. TFAE:

i) $y \in \partial f(x)$;

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Applying the same reasoning to f^* and noticing that $f^{**} = f$ if $f \in \Gamma_0$, gives the missing equivalence. $\square$

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Here's the complete picture:

Theorem 7 (Hörmander).

A function $f : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, closed and sublinear if and only if it is a support function.

Proof.

Blackboard/Notes. $\square$

Gauges and polar sets

Definition 8 (Gauge function).

Let $C \subset \mathbb{E}$. The *gauge (function)* of C is defined by $\gamma_C : x \in \mathbb{E} \mapsto \inf \{\lambda \geq 0 \mid x \in \lambda C\}$.

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Let $C \subset \mathbb{E}$. Then its *polar set* is defined by

$$C^\circ := \left\{ v \in \mathbb{E} \,\middle|\, \langle v, x \rangle \leq 1 \ (x \in C) \right\}.$$

Moreover, we put $C^{\circ\circ} := (C^\circ)^\circ$ and call it the *bipolar set* of C.

- If K is a cone then $K^\circ = \left\{ v \in \mathbb{E} \,\middle|\, \langle v, x \rangle \leq 0 \ (x \in K) \right\}$.
- For $C \subset \mathbb{E}$ we have $C^{\circ\circ} = \overline{\mathrm{conv}}\,(C \cup \{0\})$. (bipolar theorem)

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$$C^\circ := \left\{ v \in \mathbb{E} \,\middle|\, \langle v, x \rangle \leq 1 \ (x \in C) \right\}.$$

Moreover, we put $C^{\circ\circ} := (C^\circ)^\circ$ and call it the *bipolar set* of C.

- If K is a cone then $K^\circ = \left\{ v \in \mathbb{E} \,\middle|\, \langle v, x \rangle \leq 0 \ (x \in K) \right\}$.
- For $C \subset \mathbb{E}$ we have $C^{\circ\circ} = \overline{\mathrm{conv}}\,(C \cup \{0\})$. (bipolar theorem)

Proposition 10.

Let $C \subset \mathbb{E}$ be closed and convex with $0 \in C$. Then

$$\gamma_C = \sigma_{C^\circ} \overset{*}{\longleftrightarrow} \delta_{C^\circ} \quad \textit{and} \quad \gamma_{C^\circ} = \sigma_C \overset{*}{\longleftrightarrow} \delta_C.$$

Infimal projection

Theorem 11 (Infimal projection).

Let $\psi : \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ *be convex. Then the* optimal value function

$$p : \mathbb{E}_1 \to \overline{\mathbb{R}}, \; p(x) := \inf_{y \in \mathbb{E}_2} \psi(x, y)$$

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Proof.

Let $L : (x, y, \alpha) \mapsto (x, \alpha)$ and observe that

$$\mathrm{epi}_{<} p \quad = \quad \left\{ (x, \alpha) \,\middle|\, \inf_y \psi(x, y) < \alpha \right\}$$

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Hence $\mathrm{epi}_{<} p$ is a convex set, and thus p is convex. $\square$

Infimal convolution - a special case of infimal projection

Definition 12 (Infimal convolution).

Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the function

$$f \# g : \mathbb{E} \to \overline{\mathbb{R}}, \quad (f \# g)(x) := \inf_{u \in \mathbb{E}} \{f(u) + g(x - u)\}$$

is called the *infimal convolution* of f and g. We call the infimal convolution $f \# g$ *exact at* $x \in \mathbb{E}$ if

$$\operatorname*{argmin}_{u \in \mathbb{E}} \{f(u) + g(x - u)\} \neq \emptyset.$$

We simply call $f \# g$ exact if it is exact at every $x \in \operatorname{dom} f \# g$.

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We always have:

- $\operatorname{dom} f\#g = \operatorname{dom} f + \operatorname{dom} g$;
- $f\#g = g\#f$;
- f, g convex, then $f\#g$ convex (as $(f\#g)(x) = \inf_y h(x,y)$ with $h : (x,y) \mapsto f(y) + g(x-y)$ convex).

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Example 13 (Distance functions).

Let $C \subset \mathbb{E}$. Then $d_C := \delta_C \# \|\cdot\|$, i.e.

$$d_C(x) = \inf_{u \in C} \|x - u\|$$

is the distance function of C, which is hence convex if C is a convex.

Conjugacy of infimal convolution

Proposition 14 (Conjugacy of inf-convolution).

Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. *Then the following hold:*

a) $(f \# g)^* = f^* + g^*$;

b) *If* $f, g \in \Gamma_0$ *such that* $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$, *then* $(f+g)^* = \mathrm{cl}\,(f^* \# g^*)$.

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Proof.

a) For all $y \in \mathbb{E}$, we have

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b) $(f^* \# g^*)^* \overset{a)}{=} f^{**} + g^{**} \overset{f,g \in \Gamma_0}{=} f + g \overset{clear?}{\in} \Gamma$

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$$\Longrightarrow \quad \mathrm{cl}\,(f^* \# g^*) = (f^* \# g^*)^{**} = (f+g)^*.$$

$\square$

Attouch-Brézis - drop the closure!

Theorem 15 (Attouch-Brézis).

Let $f, g \in \Gamma_0$ such that

$$\mathrm{ri}\,(\mathrm{dom}\, f) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq 0 \quad (CQ).$$

Then $(f+g)^* = f^* \# g^*$, and the infimal convolution is exact, i.e. the infimum in the infimal convolution is attained on $\mathrm{dom}\, f^* \# g^*$.

Proof.

On blackboard. □

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Proof.

On blackboard. □

We note that (CQ) is always satisfied under any of the following:

- $\mathrm{int}\,(\mathrm{dom}\, f) \cap \mathrm{dom}\, g \neq \emptyset$,
- $\mathrm{dom}\, f = \mathbb{E}$,

and is equivalent to saying that

$$0 \in \mathrm{ri}\,(\mathrm{dom}\, f - \mathrm{dom}\, g).$$

Excursion: Moreau envelope and proximal operator [1]

Let $f \in \Gamma_0$ and $\lambda > 0$. Then

$$e_\lambda f := f \# \frac{1}{2\lambda}\|\cdot\|^2$$

is called the *Moreau envelope* of f.

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- $e_\lambda f \uparrow f \quad (\lambda \downarrow 0)$ (monotone pointwise convergence)
- $\operatorname{epi} e_\lambda f \to \operatorname{epi} f \quad (\lambda \downarrow 0)$ (epi-convergence)

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Conjugacy for convex-linear composites

Let $f \in \Gamma$ and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$. Then

$$Lf : \mathbb{E}' \to \overline{\mathbb{R}}, \quad (Lf)(y) := \inf \left\{ f(x) \;\middle|\; L(x) = y \right\}$$

is convex[2].

[2] Show that $\operatorname{epi}_{<} Lf = T(\operatorname{epi}_{<} f)$ for $T : (x, y) \mapsto (Tx, y)$.

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Proposition 16.

Let $g : \mathbb{E} \to \overline{\mathbb{R}}$ be proper and $L \in \mathcal{L}(\mathbb{E}, \mathbb{E}')$ and $T \in \mathcal{L}(\mathbb{E}', \mathbb{E})$. Then the following hold:

a) $(Lg)^* = g^* \circ L^*$.

b) $(g \circ T)^* = \mathrm{cl}\,(T^* g^*)$ *if* $g \in \Gamma$.

c) *The closure in b) can be dropped and the infimum is attained when finite if* $g \in \Gamma_0$ *and*

$$\mathrm{ri}\,(\mathrm{rge}\, T) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset. \tag{3}$$

Proof.

Notes and Part 2. □

[2] Show that $\mathrm{epi}_{<} Lf = T(\mathrm{epi}_{<} f)$ for $T : (x, y) \mapsto (Tx, y)$.

Infimal projection revisited

Theorem 17 (Infimal projection II).

Let $\psi \in \Gamma_0(\mathbb{E}_1 \times \mathbb{E}_2)$ and define $p : \mathbb{E}_1 \to \overline{\mathbb{R}}$ by

$$p(x) := \inf_v \psi(x, v). \tag{4}$$

Then the following hold:

a) *p is convex.*

b) *$p^* = \psi^*(\cdot, 0)$ which is closed and convex.*

c) *The condition*

$$\mathrm{dom}\, \psi^*(\cdot, 0) \neq 0 \tag{5}$$

is equivalent to having $p^* \in \Gamma_0$.

d) *If (5) holds then $p \in \Gamma_0$ and the infimum in its definition is attained when finite.*

Proof.

Blackboard/Notes. □

2. Conjugacy of composite functions via K-convexity and inf-convolution

Cone-induced ordering

Given a cone $K \subset \mathbb{E}$, the relation

$$x \leq_K y \quad :\Longleftrightarrow \quad y - x \in K \quad (x, y \in \mathbb{E})$$

induces an *ordering* on $\mathbb{E}$ which is a *partial ordering* if K is convex and pointed[3].

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- Attach to $\mathbb{E}$ a *largest element* $+\infty_\bullet$ w.r.t. $\leq_K$ which satisfies $x \leq_K +\infty_\bullet \quad (x \in \mathbb{E})$.

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- Attach to $\mathbb{E}$ a *largest element* $+\infty_\bullet$ w.r.t. $\leq_K$ which satisfies $x \leq_K +\infty_\bullet \quad (x \in \mathbb{E})$.
- Set $\mathbb{E}^\bullet := \mathbb{E} \cup \{+\infty_\bullet\}$.
- For $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ define

$$\begin{aligned} \mathrm{dom}\, F &:= \left\{x \in \mathbb{E}_1 \;\middle|\; F(x) \in \mathbb{E}_2\right\} \quad \text{(domain)},\\ \mathrm{gph}\, F &:= \{(x, F(x)) \in \mathbb{E}_1 \times \mathbb{E}_2 \mid x \in \mathrm{dom}\, F\} \quad \text{(graph)},\\ \mathrm{rge}\, F &:= \{F(x) \in \mathbb{E}_2 \mid x \in \mathrm{dom}\, F\} \quad (\textit{range}). \end{aligned}$$

[3] i.e. $K \cap (-K) = \{0\}$

K-convexity

Definition 18 (K-convexity).

Let $K \subset \mathbb{E}_2$ be a cone and $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$. Then we call F *K-convex* if

$$K\text{-epi}\, F := \left\{(x, v) \in \mathbb{E}_1 \times \mathbb{E}_2 \;\middle|\; F(x) \leq_K v\right\} \quad (K\text{-epigraph})$$

is convex (in $\mathbb{E}_1 \times \mathbb{E}_2$).

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- F is K-convex $\iff$ $F(\lambda x + (1-\lambda)y) \leq_K \lambda F(x) + (1-\lambda)F(y) \quad (x, y \in \mathbb{E}_1, \lambda \in [0, 1])$

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- F is K-convex $\iff$ $F(\lambda x + (1-\lambda)y) \preceq_K \lambda F(x) + (1-\lambda)F(y) \quad (x, y \in \mathbb{E}_1, \lambda \in [0, 1])$
- F K-convex, then $\mathrm{ri}\,(K\text{-epi}\, F) = \left\{(x, v) \;\middle|\; x \in \mathrm{ri}\,(\mathrm{dom}\, F),\ F(x) \preceq_{\mathrm{ri}\,(K)} v\right\}$

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- $K = \mathbb{R}^m_+$ and $F : \mathbb{R}^n \to (\mathbb{R}^m)^\bullet$ with $F_i \in \Gamma$ $(i = 1, \dots, m)$

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- K arbitrary, F affine.

Convexity of composite functions

For $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ we define

$$(g \circ F)(x) := \begin{cases} g(F(x)) & \text{if} \quad x \in \operatorname{dom} F, \\ +\infty & \text{else.} \end{cases}$$

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$$g(F(x)) \leq g(y) \quad ((x, y) \in K\text{-}\operatorname{epi} F) \tag{6}$$

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a) *$g \circ F$ is convex and proper.*

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Scalarization

Given $v \in \mathbb{E}_2$ and the linear form $\langle v, \cdot\rangle : \mathbb{E}_2 \to \mathbb{R}$, we set $\langle v, F\rangle := \langle v, \cdot\rangle \circ F$, i.e.

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- F is K-convex $\iff$ $\langle v, F\rangle$ is convex ($v \in -K^\circ$)
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Lemma 20 (Pennanen, JCA 1999).

Let $f \colon \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ with a convex domain and let $K \subset \mathbb{E}_2$ be the smallest closed convex cone with respect to which F is convex. Then

$$(-K)^\circ = \{v \in \mathbb{E}_2 \mid \langle v, F\rangle \text{ is convex}\}.$$

The main result

Theorem 21 (Conjugacy for composite function, H./Nguyen '19, Bot et. al '11).

Let $K \subset \mathbb{E}_2$ be a closed convex cone, $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ K-convex such that K-$\mathrm{epi}\, F$ is closed and $g_0 \in \Gamma(\mathbb{E}_2)$ such that (6) *is satisfied, i.e.*

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Under the CQ

$$F(\mathrm{ri}\,(\mathrm{dom}\, F)) \cap \mathrm{ri}\,(\mathrm{dom}\, g - K) \neq \emptyset \tag{7}$$

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$$(g \circ F)^*(p) = \min_{v \in -K^\circ} g^*(v) + \langle v, F\rangle^*(p)$$

with $\mathrm{dom}\,(g \circ F)^* = \left\{p \in \mathbb{E}_1 \;\middle|\; \exists v \in \mathrm{dom}\, g^* \cap (-K^\circ) : \langle v, F\rangle^*(p) < +\infty\right\}.$

Proof.

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Extension to the additive composite setting

Corollary 22 (Conjugate of additive composite functions, H./Nguyen '19).

Under the assumptions of Theorem 21 let $f \in \Gamma_0$ *such that*

$$F(\mathrm{ri}\,(\mathrm{dom}\, f \cap \mathrm{dom}\, F)) \cap \mathrm{ri}\,(\mathrm{dom}\, g - K) \neq \emptyset. \tag{8}$$

Then

$$(f + g \circ F)^*(p) = \min_{\substack{v \in -K^\circ, \\ y \in \mathbb{E}_1}} g^*(v) + f^*(y) + \langle v, F\rangle^*(p - y).$$

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(Sketch) Apply Theorem 21 to $\tilde{g} : (s, y) \in \mathbb{R} \times \mathbb{E}_2 \mapsto s + g(y)$, $\tilde{F} : x \in \mathbb{E}_1 \to (f(x), x)$ and $\tilde{K} := \mathbb{R}_+ \times K$. $\square$

The case $K = -\operatorname{hzn} g$

For $g \in \Gamma_0$ its *horizon function* g^∞ is given via

$$\operatorname{epi} g^\infty = (\operatorname{epi} g)^\infty.$$

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Corollary 23 (Burke ’91, H./Nguyen ’19).

Let $g \in \Gamma_0(\mathbb{E}_2)$ and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be $(-\mathrm{hzn}\, g)$-convex with $-\mathrm{hzn}\, g$-epi F closed such that

$$F(\mathrm{ri}\,(\mathrm{dom}\, F)) \cap \mathrm{ri}\,(\mathrm{dom}\, g + \mathrm{hzn}\, g) \ne \emptyset.$$

Then

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The linear case

Corollary 24 (The linear case).

Let $g \in \Gamma(\mathbb{E}_2)$ *and* $F : \mathbb{E}_1 \to \mathbb{E}_2$ *linear such that*

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Proof.

We notice that F is $\{0\}$-convex. Hence we can apply Theorem 21 with $K = \{0\}$. Condition (7) then reads $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$, which is our assumption. Hence we obtain

$$(g \circ F)^*(p) = \min_{v \in -K^\circ} g^*(v) + \langle v, F \rangle^*(p) = \min_{v \in \mathbb{E}_2} g^*(v) + \delta_{\{F^*(v)\}}(p).$$

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Conic programming duality

Consider the general conic program

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Conic programming duality

Consider the general conic program

$$\min f(x) \quad \text{s.t.} \quad F(x) \in -K \tag{9}$$

or equivalently

$$\min_{x\in\mathbb{E}_1} f(x) + (\delta_{-K} \circ F)(x) \tag{10}$$

where $f : \mathbb{E}_1 \to \mathbb{R}$ is convex, $F : \mathbb{E}_1 \to \mathbb{E}_2$ is K-convex and $K \subset \mathbb{E}_2$ is a closed, convex cone. The qualification condition (7) turns into a *generalized Slater condition*

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Theorem 25 (Strong duality and dual attainment for conic programming).

Let $f : \mathbb{E}_1 \to \mathbb{R}$ is convex, $K \subset \mathbb{E}_2$ a closed, convex cone, and let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be K-convex with closed K-epigraph. If (11) *holds then*

$$\inf_{x\in\mathbb{E}_1} f(x) + (\delta_{-K} \circ F)(x) = \max_{v\in -K^\circ} -f^*(y) - (\delta_{-K} \circ F)^*(-y) = \max_{v\in -K^\circ} \inf_{x\in\mathbb{E}_1} f(x) + \langle v, F(x)\rangle .$$

Conjugate of pointwise maximum of convex functions

Proposition 26.

For $f_1, \dots, f_m \in \Gamma_0(\mathbb{E})$ define $f := \max_{i=1,\dots,m} f_i$. Then $f \in \Gamma_0(\mathbb{E})$ with

$$f^*(x) = \min_{v \in \Delta_m} \left(\sum_{i=1}^{m} v_i f_i \right)^* (x).$$

[4] $\Delta_m = \left\{ \lambda \in \mathbb{R}^m \mid \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0 \ (i1, \dots, m) \right\}$

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$$F : x \mapsto \begin{cases} (f_1(x), \ldots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom} f_i, \\ +\infty_\bullet & \text{otherwise,} \end{cases} \quad \text{and} \quad g : y \mapsto \max_{i=1,\ldots,m} x_i.$$

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3. A new class of matrix support functionals

Motivation I: Nuclear norm minimization/smoothing

Rank minimization (→ Netflix recommender problem)

$$\min_{X \in \mathbb{R}^{n\times m}} \operatorname{rank} X \quad \text{s.t.} \quad MX = B \quad (M \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{p\times m}) \tag{12}$$

[5] $\sigma(X) = (\sigma_1, \dots, \sigma_n)$ is the vector of positive singular values of X.

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- Approximating the rank function (→ combinatorial)

$$\operatorname{rank} X = \|\sigma(X)\|_0 \overset{\text{Convex approx.}}{\sim} \|\sigma(X)\|_1 =: \|X\|_* \quad (\textit{nuclear norm})^5$$

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Motivation II: Maximum likelihood estimation

Let $y_i \in \mathbb{R}^n$ $(i = 1, \ldots, N)$ be measurements of

$$y \sim N(\mu, \Sigma) \quad (\mu \in \mathbb{R}^n, \Sigma \in \mathbb{S}^n_{++} \to \text{unknown})$$

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- Likelihood function:

$$\ell(\mu, \Sigma) := \frac{1}{(2\pi)^{n/2}} \prod_{i=1}^{N} \frac{1}{(\det \Sigma)^{1/2}} \exp\left(-\frac{1}{2}(y_i - \mu)^T \Sigma^{-1} (y_i - \mu)\right)$$

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$$\max_{(\mu,\Sigma)} \ell(\mu, \Sigma) \quad \Leftrightarrow \quad \min_{(\mu,\Sigma)} -\log \ell(\mu, \Sigma)$$

$$\overset{x_i := y_i - \mu}{\Leftrightarrow} \quad \min_{(X,\Sigma) \in \mathbb{R}^{n\times N} \times \mathbb{S}^n_{++}} \frac{1}{2} \operatorname{tr}\left(X^T \Sigma^{-1} X\right) + \frac{N}{2} \log(\det \Sigma)$$

The Moore-Penrose pseudoinverse

Theorem 27 (Moore-Penrose pseudoinverse).

Let $A \in R^{m\times n}$ with $\mathrm{rank}\, A = r$ and the singular value decomposition

$$A = U\Sigma V^T \quad \text{with} \quad \Sigma = \mathrm{diag}\,(\sigma_i), \quad U, V \text{ orthogonal.}$$

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The matrix

$$A^\dagger := V\Sigma^\dagger U^T \quad \text{with} \quad \Sigma^\dagger := \begin{pmatrix} \sigma_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \sigma_r^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

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Moreover:

c) A invertible $\quad\Rightarrow\quad A^\dagger = A^{-1}$

d) $A \succ 0 \quad\Rightarrow\quad A^\dagger \succ 0$

The closure of the matrix-fractional function

Put $\mathbb{E} := \mathbb{R}^{n\times m} \times \mathbb{S}^n$.

$$\phi : (X, V) \in \mathbb{E} \mapsto \begin{cases} \frac{1}{2}\mathrm{tr}\,(X^T V^{-1} X) & \text{if} \quad V \succ 0, \\ +\infty & \text{else.} \end{cases} \qquad \text{(matrix-fractional function)}$$

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$$\overset{\text{Schur}}{\Rightarrow} \qquad \mathrm{epi}\,\phi = \left\{(X, V, \alpha) \mid \exists\, Y \in \mathbb{S}^m : \begin{pmatrix} V & X \\ X^T & Y \end{pmatrix} \succeq 0,\ V \succ 0,\ \frac{1}{2}\mathrm{tr}\,(Y) \le \alpha \right\}$$

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$\overset{\text{Hörmander's Theorem}}{\Rightarrow}$ $\mathrm{cl}\,\phi$ is a support function

Motivation III: Quadratic programming

For $A \in \mathbb{R}^{p\times n}$ and $V \in \mathbb{S}^n$ put

$$M(V) := \begin{pmatrix} V & A^T \\ A & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{K}_A := \left\{ V \in \mathbb{S}^n \;\middle|\; u^T V u \geq 0 \ (u \in \ker A) \right\}.$$

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Theorem 28 (Burke, H. '15).

For $b \in \operatorname{rge} A$, we have

$$\inf_{u\in\mathbb{R}^n} \left\{ \frac{1}{2} u^T V u - x^T u \mid Au = b \right\} = \begin{cases} -\frac{1}{2}\begin{pmatrix} x \\ b \end{pmatrix}^T M(V)^\dagger \begin{pmatrix} x \\ b \end{pmatrix} & \text{if} \quad x \in \operatorname{rge}[V\ A^T],\ V \in \mathcal{K}_A, \\ -\infty & \text{else.} \end{cases}$$

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Question: For $A \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{p\times m}$, is

$$\varphi_{A,B} : (X, V) \in \mathbb{E} \mapsto \begin{cases} \frac{1}{2} \mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B} \right) & \text{if} \ \ \mathrm{rge}\,\binom{X}{B} \subset \mathrm{rge}\, M(V),\ V \in \mathcal{K}_A, \\ +\infty & \text{else} \end{cases}$$

a support function?

A new class of matrix support functions

Define

$$\mathcal{D}(A,B) := \left\{ \left(Y, -\frac{1}{2} YY^T \right) \in \mathbb{E} \;\middle|\; Y \in \mathbb{R}^{n \times m} : \; AY = B \right\} \quad (A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{p \times m}).$$

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Theorem 29 (Burke, H. '15).

For $\mathrm{rge}\, B \subset \mathrm{rge}\, A$

$$\sigma_{\mathcal{D}(A,B)}(X,V) = \begin{cases} \frac{1}{2}\mathrm{tr}\left(\binom{X}{B}^T M(V)^\dagger \binom{X}{B} \right) & \text{if } \; \mathrm{rge}\, \binom{X}{B} \subset \mathrm{rge}\, M(V), \;\; V \in \mathcal{K}_A, \\ +\infty & \text{else} \end{cases} \qquad ((X,V) \in \mathbb{E})$$

with

$$\mathrm{int}\,(\mathrm{dom}\, \sigma_{D(A,B)}) = \{ (X,V) \in \mathbb{E} \mid V \in \mathrm{int}\, \mathcal{K}_A \}.$$

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with

$$\mathrm{int}\,(\mathrm{dom}\, \sigma_{D(A,B)}) = \{(X,V) \in \mathbb{E} \mid V \in \mathrm{int}\, \mathcal{K}_A\}.$$

In particular,

$$\sigma_{\mathcal{D}(0,0)}(X,V) = \begin{cases} \frac{1}{2}\mathrm{tr}\,(X^T V^\dagger X) & \text{if} \quad V \succeq 0, \mathrm{rge}\, X \subset \mathrm{rge}\, V, \\ +\infty & \text{else} \end{cases} \quad = \mathrm{cl}\, \phi(X,V).$$

Proof.

Blackboard/Notes. □

Closed convex hull of $\mathcal{D}(A, B)$: Carathéodory-based description

Recall

$$\partial\sigma_{\mathcal{D}(A,B)}(X, V) = \left\{(Y, W) \in \overline{\mathrm{conv}}\,\mathcal{D}(A, B) \;\middle|\; (X, V) \in N_{\overline{\mathrm{conv}}\,\mathcal{D}(A,B)}(Y, W)\right\} \quad \text{and} \quad \sigma_{\mathcal{D}(A,B)} = \delta^*_{\overline{\mathrm{conv}}\,\mathcal{D}(A,B)}$$

where

$$\mathcal{D}(A, B) := \left\{\left(Y, -\frac{1}{2}YY^T\right) \in \mathbb{E} \;\middle|\; Y \in \mathbb{R}^{n\times m} : \; AY = B\right\}.$$

[6] $d \otimes I_m = (d_i I_m) \in \mathbb{R}^{m(\kappa+1)}$
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Proposition 30 (Burke, H. '15).

$$\overline{\mathrm{conv}}\, \mathcal{D}(A, B) = \left\{(Z(d \otimes I_m), -\frac{1}{2}ZZ^T) \;\middle|\; (d, Z) \in \mathcal{F}(A, B)\right\}.^6$$

where

$$\mathcal{F}(A, B) := \left\{(d, Z) \in \mathbb{R}^{\kappa+1} \times \mathbb{R}^{n\times m(\kappa+1)} \;\middle|\; \begin{array}{l} d \geq 0, \; \|d\| = 1, \\ AZ_i = d_i B \; (i = 1, \dots, \kappa + 1) \end{array}\right\}.^7$$

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[6] $d \otimes I_m = (d_i I_m) \in \mathbb{R}^{m(\kappa+1)}$

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Closed convex hull of $\mathcal{D}(A, B)$: A new description

Define

$$\Omega(A, B) := \left\{ (Y, W) \in \mathbb{E} \;\middle|\; AY = B \;\text{ and }\; \frac{1}{2}YY^T + W \in \mathcal{K}_A^\circ \right\}, \tag{13}$$

and observe that

$$\mathcal{K}_A^\circ = \mathbb{R}_+ \mathrm{conv}\left\{ -vv^T \mid v \in \ker A \right\}.$$

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Theorem 31 (Burke, Gao, H. '17).

We have

$$\overline{\mathrm{conv}}\, \mathcal{D}(A,B) = \Omega(A,B).$$

In particular,

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Proof.

Notes. □

Corollary 32 (Conjugate of GMF).

We have

$$\sigma^*_{\mathcal{D}(A,B)} = \delta_{\Omega(A,B)}.$$

Convex geometry of $\Omega(A, B)$

Recall that $\Omega(A,B) := \left\{(Y,W) \in \mathbb{E} \;\middle|\; AY = B \text{ and } \frac{1}{2}YY^T + W \in \mathcal{K}_A^\circ\right\}$

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Let $\Omega(A, B)$ be given as above. Then:

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Proposition 34 (Burke, Gao, H. '17).

Let $\Omega(A,B)$ be given as above and let $(Y,W) \in \Omega(A,B)$. Then

$$N_{\Omega(A,B)}(Y,W) = \left\{(X,V) \in \mathbb{E} \;\middle|\; \begin{array}{l} V \in \mathcal{K}_A,\ \left\langle V, \dfrac{1}{2}YY^T + W\right\rangle = 0 \\ \text{and } \operatorname{rge}(X - VY) \subset (\ker A)^\perp \end{array}\right\}.$$

Subdifferentiation of the GMF

For any set C recall that

$$\partial\sigma_C(x) = \left\{z \in \overline{\mathrm{conv}}\, C \;\middle|\; x \in N_{\overline{\mathrm{conv}}\, C}(z)\right\} \tag{14}$$

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Corollary 35 (The subdifferential of $\sigma_{\mathcal{D}(A,B)}$).

For all $(X, V) \in \mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}$, *we have*

$$\partial\sigma_{\mathcal{D}(A,B)} = \left\{ (Y, W) \in \Omega(A, B) \;\middle|\; \begin{array}{l} \exists Z \in \mathbb{R}^{p\times m} : \; X = VY + A^T Z, \\ \left\langle V, \, \frac{1}{2} YY^T + W \right\rangle = 0 \end{array} \right\}.$$

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Corollary 36.

The GMF $\sigma_{\mathcal{D}(A,B)}$ *is (continuously) differentiable on the interior of its domain with*

$$\nabla\sigma_{\mathcal{D}(A,B)}(X, V) = \left(Y, -\frac{1}{2} YY^T \right) \quad ((X, V) \in \mathrm{int}\,(\mathrm{dom}\, \sigma_{\mathcal{D}(A,B)}))$$

where $Y := A^{\dagger} B + (P(P^T VP)^{-1} P^T)(X - A^{\dagger} X)$, $P \in \mathbb{R}^{n \times (n-p)}$ *is any matrix whose columns form an orthonormal basis of* $\ker A$ *and* $p := \mathrm{rank}\, A$.

Conjugate of variational Gram functions

For $M \subset \mathbb{S}^n_+$ (w.lo.g.) closed, convex, the associated *variational Gram function (VGF)* is given by

$$\Omega_M : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{+\infty\}, \quad \Omega_M(X) = \frac{1}{2}\sigma_M(XX^T).$$

With

$$F : \mathbb{R}^{n\times m} \to \mathbb{S}^n, \quad F(X) = \frac{1}{2}XX^T. \tag{15}$$

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- $\mathbb{S}^n_+$ is the smallest closed convex cone in $\mathbb{S}^n$ with respect to which F is convex;
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Theorem 37 (Jalali et al. '17/ Burke, Gao, H. '19).

Let $M \subset \mathbb{S}^n_+$ be nonempty, convex and compact. Then Ω^*_M is finite-valued and given by

$$\Omega^*(X) \quad = \quad \frac{1}{2}\min_{V\in M}\left\{\mathrm{tr}\,(X^TV^\dagger X) \;\middle|\; \mathrm{rge}\,X \subset \mathrm{rge}\,V\right\}.$$

Proof.

Blackboard/Notes. □

Nuclear norm smoothing

For $A \in \mathbb{R}^{p\times n}$ set

$$\mathrm{Ker}\, A := \{V \in \mathbb{R}^{n\times n} \mid AV = 0\} \quad \text{and} \quad \mathrm{Rge}\, A := \left\{W \in \mathbb{R}^{n\times n} \;\middle|\; \mathrm{rge}\, W \subset \mathrm{rge}\, A\right\}.$$

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Theorem 38.

Let $p : \mathbb{R}^{n \times m} \to \overline{\mathbb{R}}$ be defined by

$$p(X) = \inf_{V \in \mathbb{S}^n} \sigma_{\Omega(A,0)}(X, V) + \left\langle \bar{U},\, V \right\rangle$$

for some $\bar{U} \in \mathbb{S}^n_+ \cap \mathrm{Ker}\, A$ and set $C(\bar{U}) := \left\{Y \;\middle|\; \frac{1}{2} YY^T \preceq \bar{U}\right\}$. Then we have:

a) *$p^* = \delta_{C(\bar{U}) \cap \mathrm{Ker}\, A}$ is closed, proper, convex.*

b) *$p = \sigma_{C(\bar{U}) \cap \mathrm{Ker}\, A} = \gamma_{C(\bar{U})^\circ + \mathrm{Rge}\, A^T}$ is sublinear, finite-valued, nonnegative and symmetric (i.e. a seminorm).*

c) *If $\bar{U} \succ 0$ with $2\bar{U} = LL^T$ ($L \in \mathbb{R}^{n \times n}$) and $A = 0$ then $p = \sigma_{C(\bar{U})} = \|L^T(\cdot)\|_*$, i.e. p is a norm with $C(\bar{U})^\circ$ as its unit ball and $\gamma_{C(\bar{U})}$ as its dual norm.*

Proof.

Blackboard/Notes. □

Current and future directions

- K-convexity
  - When is $\overline{\mathrm{conv}}\,(\mathrm{gph}\,F) = K\text{-}\mathrm{epi}\,F$ for F K-convex ?

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- <u>Generalized matrix-fractional function</u>
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$$p(X) = \inf_{V \in \mathbb{S}^n} \sigma_{\Omega(A,B)}(X, V) + h(V).$$

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