Cone-convexity and composite functions

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joint work with

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1. Fundamentals from Convex Analysis

Convex sets and cones

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$S\subset\mathbb{E}$ is said to be

- *convex* if $\lambda S+(1-\lambda)S\subset S \quad (\lambda\in(0,1))$;
- a *cone* if $\lambda S\subset S \quad (\lambda\geq 0)$;
- For a cone K its *polar* is $K^\circ=\left\{v \,\middle|\, \langle v,x\rangle\leq 0\ (x\in K)\right\}$.

Note that $K\subset\mathbb{E}$ is a *convex cone* iff $K+K\subset K$.

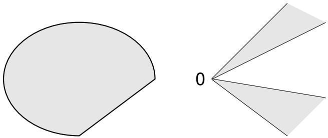

Figure: Convex set/non-convex cone

The topology relative to the affine hull

Affine set: A set $S = U + x$ with $x \in \mathbb{E}$ and a subspace $U \subset$ is called *affine*. This is characterized by

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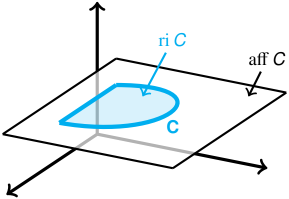

C	$\operatorname{aff} C$	$\operatorname{ri} C$
$\{x\}$	$\{x\}$	$\{x\}$
$[x, x']$	$\{\lambda x + (1 - \lambda)x' \mid \lambda \in \mathbb{R}\}$	(x, x')
$\overline{B}_\varepsilon(x)$	$\mathbb{E}$	$B_\varepsilon(x)$

Table: Examples for relative interiors

Extended real-valued functions: An epigraphical perspective

Let $f : \mathbb{E} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

- $\operatorname{epi} f := \left\{(x, \alpha) \in \mathbb{E} \times \mathbb{R} \,\middle|\, f(x) \le \alpha\right\}$ (epigraph)

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$\to$ f is uniquely determined through $\operatorname{epi} f$!

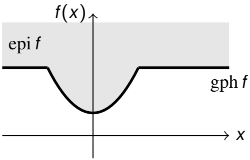

Figure: Epigraph of $f : \mathbb{R} \to \mathbb{R}$

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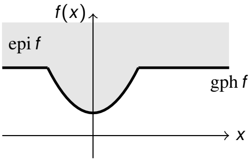

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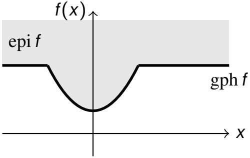

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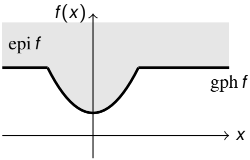

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f convex	$:\Leftrightarrow$	$\operatorname{epi} f / \operatorname{epi}_{<} f$ convex	$\Leftrightarrow$ [1]	$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ $(x, y \in \mathbb{E},\ \lambda \in [0,1])$
f lsc (closed)	$:\Leftrightarrow$	$\operatorname{epi} f$ closed	$\Leftrightarrow$	$(\operatorname{cl} f)(x) := \liminf_{k\to\infty} f(x_k) \ge f(x)$ $(\{x_k\} \to x,\ x \in \mathbb{E})$

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The convex subdifferential

Definition 1.

Let $f : \mathbb{E} \to \overline{\mathbb{R}}$. A vector $v \in \mathbb{E}$ is called a *subgradient* of v at $\bar{x}$ if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle \quad (x \in \mathbb{E}). \tag{1}$$

We denote by $\partial f(\bar{x})$ the set of all subgradients of f at $\bar{x}$ and call it the *(convex) subdifferential* of f at $\bar{x}$.

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- $\mathrm{ri}\,(\mathrm{dom}\, f) \subset \mathrm{dom}\, \partial f \subset \mathrm{dom}\, f$ (f convex).

The Fenchel conjugate

Definition 2 (Fenchel conjugate).

Let $f : \mathbb{E} \to \overline{\mathbb{R}}$ proper. The function $f^* : \mathbb{E} \to \overline{\mathbb{R}}$ defined through

$$f^*(v) = \sup_{x \in \mathbb{E}} \{\langle v, x\rangle - f(x)\} \quad (v \in \mathbb{E})$$

is called the (Fenchel) conjugate of f. The function $(f^{**}) := (f^*)^*$ is called the biconjugate of f.

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Theorem 3 (Interplay of subdifferential and conjugate).

Let f lsc, proper, convex. TFAE:

i) $y \in \partial f(x)$;

ii) $f(x) + f^*(y) = \langle x, y \rangle$;

iii) $x \in \partial f^*(y)$.

In particular, $\partial f^* = (\partial f)^{-1}$.

Infimal convolution

Definition 4 (Infimal convolution).

Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$. Then the function

$$f \# g : \mathbb{E} \to \overline{\mathbb{R}}, \quad (f \# g)(x) := \inf_{u \in \mathbb{E}} \{f(u) + g(x - u)\}$$

is called the *infimal convolution* of f and g. We call the infimal convolution $f \# g$ *exact at* $x \in \mathbb{E}$ if

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We simply call $f \# g$ exact if it is exact at every $x \in \operatorname{dom} f \# g$.

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Theorem 5 (Conjugacy of inf-convolution).

Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be proper and convex. Then the following hold:

a) $(f \# g)^* = f^* + g^*$;

b) *If* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *then* $(\operatorname{cl} f + \operatorname{cl} g)^* = \operatorname{cl}(f^* \# g^*)$.

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c) *If*

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq 0 \quad (CQ),$$

the closures in b) are superfluous and the infimum is attained for all $y \in \operatorname{dom}(f^* \# g^*)$, *i.e.*

$$(f+g)^*(y) = \min_{u \in \mathbb{E}} \{f^*(u) + g^*(y-u)\}.$$

2. K-Convexity and Convex Convex-Composite Functions

Cone-induced ordering

Given a cone $K \subset \mathbb{E}$, the relation

$$x \leq_K y \quad :\Longleftrightarrow \quad y - x \in K \quad (x, y \in \mathbb{E})$$

induces an *ordering* on $\mathbb{E}$ which is a *partial ordering* if K is convex and pointed[1].

[1] i.e. $K \cap (-K) = \{0\}$

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- K arbitrary, F affine.

Composition and scalarization

For $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ we define

$$(g \circ F)(x) := \begin{cases} g(F(x)) & \text{if} \quad x \in \operatorname{dom} F, \\ +\infty & \text{else.} \end{cases}$$

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Lemma 7 (Scalarization).

Let $F:\mathbb{E}_1\to\mathbb{E}_2^\bullet$ and let $K\subset\mathbb{E}_2$ be a closed, convex cone. Then the following hold for all $(u,v)\in\mathbb{E}_1\times\mathbb{E}_2$:

a) *F is K-convex if and only if $\langle v, F\rangle$ is convex for all $v\in -K^\circ$.*

b) $\sigma_{K\text{-epi}\,F}(u,v)=\sigma_{\operatorname{gph}F}(u,v)+\delta_{K^\circ}(v)$.

c) $\sigma_{\operatorname{gph}F}(u,-v)=\langle v, F\rangle^*(u)$.

d) *If F is linear then $\langle v, F\rangle^*=\delta_{\{F^*(v)\}}$.*

Here, for $S\subset\mathbb{E}$ we define

$$\delta_S: x\in\mathbb{E}\mapsto\begin{cases}0, & \in S,\\ +\infty, & x\notin S\end{cases}\quad\text{and}\quad \sigma_S:=\delta_S^*: y\mapsto\sup_{x\in S}\langle x, y\rangle.$$

Convexity and closedness of compositions

Proposition 8 (Combari et al. '95/Pennanen '99/Bot et al. '08/Burke, H., Nguyen '19).

Let $K \subset \mathbb{E}_2$ be a convex cone, $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ K-convex and $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ convex and proper such that $\mathrm{rge}\, F \cap \mathrm{dom}\, g \neq \emptyset$. If

$$g(F(x)) \leq g(y) \quad ((x, y) \in K\text{-epi}\, F) \tag{2}$$

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Observe that (2) is strictly weaker than (3)!

The main result

Theorem 9 (Burke,H., Nguyen '19).

Let $K \subset \mathbb{E}_2$ be a closed, convex cone, $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ K-convex and $f : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$, $g : \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}$ proper, convex such that

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Then ($f + g \circ F$ is convex and) the following hold:

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b) $\partial(f + g \circ F)(\bar{x}) = \partial f(\bar{x}) + \bigcup_{v \in \partial g(F(\bar{x}))} \partial \langle v, F\rangle(\bar{x}) \quad (\bar{x} \in \mathrm{dom}\, f \cap \mathrm{dom}\, g \circ F).$

- The condition (2) that g be increasing w.r.t to K-epi F is satisfied if g is K-increasing.
- The CQ (5) is trivially satisfied when f and g are finite-valued (and F is proper).

Sketch of the proof with $f \equiv 0$

Define $\phi\colon \mathbb{E}_1 \times \mathbb{E}_2 \to \mathbb{R} \cup \{+\infty\}\colon (x, y) \mapsto g(y)$ and observe that

$$\mathrm{ri}\,(\mathrm{dom}\,\phi) \cap \mathrm{ri}\,(\mathrm{dom}\,\delta_{K\text{-}\mathrm{epi}\,F}) \neq \emptyset \quad \Longleftrightarrow \quad F(\mathrm{ri}\,(\mathrm{dom}\,F)) \cap \mathrm{ri}\,(\mathrm{dom}\,g - K) \neq \emptyset \quad \text{(CQ)}.$$

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$$(g \circ F)^*(p) = \sup_{x \in \mathbb{E}_1} \{\langle x, p \rangle - g(F(x))\}$$

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\end{aligned}$$

For the subdifferential formula use the Fenchel-Young inequality and Theorem 3, respectively.

The case $K = -\operatorname{hzn} g$

For $\phi : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ lsc, proper, convex we define its *horizon cone* by

$$\operatorname{hzn} \phi := \{v \mid \forall t \geq 0 : \ x + tv \in \operatorname{lev}_\alpha \phi\}^2,$$

where $\operatorname{lev}_\alpha \phi$ is any nonempty sublevel set of ϕ.

[2] I.e. $\operatorname{hzn} \phi$ is the closed, convex (horizon) cone $(\operatorname{lev}_\alpha \phi)^\infty$.

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Lemma 10 (Burke, H., Nguyen '19).

Let $g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc, proper, convex. Then g is $(-\operatorname{hzn} g)$-increasing.

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Corollary 11 (Burke, H., Nguyen '19).

$g : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ be lsc, proper, convex and let $F : \mathbb{E}_1 \to \mathbb{E}_2^\bullet$ be $(-\mathrm{hzn}\, g)$-convex such that

$$F(\mathrm{ri}\,(\mathrm{dom}\, F)) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset.$$

Then

$$(g \circ F)^*(p) = \min_{v \in \mathbb{E}_2} g^*(v) + \langle v, F \rangle^* (p)$$

and

$$\partial (g \circ F)(\bar{x}) = \bigcup_{v \in \partial g(F(\bar{x}))} \partial \langle v, F \rangle (\bar{x}) \quad (\bar{x} \in \mathrm{dom}\, g \circ F).$$

Proof.

$K = -\mathrm{hzn}\, g$ and observe that $\mathrm{dom}\, g^* \subset \overline{\mathrm{cone}}\,(\mathrm{dom}\, g^*) = (\mathrm{hzn}\, g)^\circ$. □

[2] I.e. $\mathrm{hzn}\, \phi$ is the closed, convex (horizon) cone $(\mathrm{lev}_\alpha \phi)^\infty$.

Component-wise convex functions

Corollary 12.

Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ *be proper, convex and* $\mathbb{R}^m_+$*-increasing,* $F : \mathbb{E} \to (\mathbb{R}^m)^\bullet$ *with* F_i $(i = 1, \dots, m)$ *proper, convex such that*

$$F\left(\bigcap_i^m \mathrm{ri}\,(\mathrm{dom}\, F_i)\right) \cap \mathrm{ri}\,(\mathrm{dom}\, g) \neq \emptyset.$$

Then

$$(g \circ F)^*(p) = \min_{v \geq 0} g^*(v) + \left(\sum_{i=1}^m v_i F_i\right)^* (p)$$

and

$$\partial(g \circ F)(\bar{x}) = \bigcup_{v \in \partial g(F(\bar{x}))} \sum_{i=1}^m v_i \partial F_i(\bar{x}) \quad (\bar{x} \in \mathrm{dom}\, g \circ F).$$

Proof.

Use $K = \mathbb{R}^m_+$ and observe that F is K-convex. □

3. Applications

Conic programming

Consider the general conic program

$$\min f(x) \quad \text{s.t.} \quad F(x) \in -K \tag{6}$$

with

- $K \subset \mathbb{E}_2$ a closed, convex cone,
- $F : \mathbb{E}_1 \to \mathbb{E}_2$ K-convex,
- $f : \mathbb{E}_1 \to \mathbb{R} \cup \{+\infty\}$ proper and convex.

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Problem (6) can be written in the additive composite form

$$\min_{x \in \mathbb{E}_1} f(x) + (\delta_{-K} \circ F)(x).$$

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Then $g = \delta_{-K}$ is K-increasing and qualification condition (5) reads

$$F(\mathrm{ri}\,(\mathrm{dom}\, f)) \cap \mathrm{ri}\,(-K) \neq \emptyset. \tag{7}$$

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Under (7) we have

$$\begin{aligned} \inf_{x \in \mathbb{E}_1} f(x) + (\delta_{-K} \circ F)(x) &= \max_{v \in -K^\circ} -f^*(y) - (\delta_{-K} \circ F)^*(-y) \\ &= \max_{v \in -K^\circ} \inf_{x \in \mathbb{E}_1} f(x) + \langle v, F(x) \rangle. \end{aligned}$$

Conic programming II

Consider again

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Observe that

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Theorem 13 (Pennanen '99/Burke, H.,Nguyen '19).

Let $f \in \Gamma(\mathbb{E}_1)$, $K \subset \mathbb{E}_2$ a closed, convex cone, and let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be K-convex. Then the condition

$$0 \in \partial f(\bar{x}) + \bigcup_{v \in N_{-K}(F(\bar{x}))} \partial \langle v, F\rangle (\bar{x}) \tag{9}$$

is sufficient for $\bar{x}$ to be a minimizer of (8). Under the condition $F(\mathrm{ri}\,(\mathrm{dom}\, f)) \cap \mathrm{ri}\,(-K) \neq \emptyset$ it is also necessary.

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Corollary 14.

Let $f : \mathbb{E}_1 \to \mathbb{R}$ be differentiable and convex, $K \subset \mathbb{E}_2$ a closed, convex cone, and let $F : \mathbb{E}_1 \to \mathbb{E}_2$ be differentiable and K-convex. Then the condition

$$-\nabla f(\bar{x}) \in F'(\bar{x})^* N_{-K}(F(\bar{x})) \tag{10}$$

is sufficient for $\bar{x}$ to be a minimizer of (8). Under the condition $\mathrm{rge}\, F \cap \mathrm{ri}\,(-K) \neq \emptyset$ it is also necessary.

Variational Gram functions

Given $M \subset \mathbb{S}^n_+$ closed and convex, the associated *variational Gram function (VGF)* is given by

$$\Omega_M : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{+\infty\}, \quad \Omega_M(X) = \frac{1}{2}\sigma_M(XX^T) := \sup_{V\in M} \operatorname{tr}(VXX^T).$$

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- σ_M is $\mathbb{S}^n_+$-increasing;
- M is bounded if and only if $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom}\sigma_M) \neq \emptyset$.

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$$\Omega_M : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{+\infty\}, \quad \Omega_M(X) = \frac{1}{2}\sigma_M(XX^T) := \sup_{V\in M} \operatorname{tr}(VXX^T).$$

Then

- $F : X \in \mathbb{R}^{n\times m} \mapsto XX^T$ is $\mathbb{S}^n_+$-convex;
- $\sigma_{K\text{-epi}\,F}(X,-V) = \begin{cases} \frac{1}{2}\operatorname{tr}\left(X^T V^\dagger X\right), & \text{if} \quad \operatorname{rge} X \subset \operatorname{rge} V,\ V \succeq 0, \\ +\infty, & \text{else} \end{cases}$ (matrix-fractional function);
- σ_M is $\mathbb{S}^n_+$-increasing;
- M is bounded if and only if $\operatorname{rge} F \cap \operatorname{ri}(\operatorname{dom}\sigma_M) \neq \emptyset$.

Proposition 15 (Jalali et al. '17/Burke, Gao, H.' 18/Burke, H.,Nguyen '19).

Let $M \subset \mathbb{S}^n_+$ be nonempty, convex and compact. Then Ω_M^* is finite-valued and given by

$$\Omega_M^*(X) = \frac{1}{2}\min_{V\in M}\left\{\operatorname{tr}(X^T V^\dagger X) \;\middle|\; \operatorname{rge} X \subset \operatorname{rge} V\right\}.$$

and

$$\partial\Omega_M(X) = \left\{VX \;\middle|\; V \in \arg\max_M \left\langle XX^T, \cdot\right\rangle\right\}$$

is compact for all X.

Extending the matrix-fractional function

Consider the Euclidean space $\mathbb{G} := \mathbb{C}^{n\times m} \times \mathbb{H}^n$ equipped with the inner product

$$\langle\cdot,\cdot\rangle : ((X,U),(Y,V)) \in \mathbb{G}\times\mathbb{G} \mapsto \mathrm{Re}\,\mathrm{tr}\,(Y^*X) + \mathrm{Re}\,\mathrm{tr}\,(VU).$$

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Define the mapping $F : \mathbb{G} \to (\mathbb{H}^n)^\bullet$ by

$$F(X, V) := \begin{cases} X^*V^\dagger X & \text{if} \quad \mathrm{rge}\, X \subset \mathrm{rge}\, V, \\ +\infty_\bullet, & \text{else}, \end{cases} \tag{11}$$

where X^* is the adjoint of X and $V^\dagger$ is the Moore-Penrose pseudoinverse of V.

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Proposition 16 (Burke, H., Nguyen, '11).

Let $F : \mathbb{G} \to (\mathbb{H}^n)^\bullet$ be given by (11) *and define $\tilde{\gamma} : \mathbb{G} \to \mathbb{R} \cup \{+\infty\}$ by*

$$\tilde{\gamma}(X,V) := \begin{cases} \frac{1}{2}\mathrm{tr}\,(F(X,V)), & \text{if} \quad \mathrm{rge}\,X \subset \mathrm{rge}\,V,\ V \succeq 0 \\ +\infty, & \text{else}. \end{cases}$$

Then $\tilde{\gamma}$ is lsc, proper, and convex, hence a support function.

Spectral Functions

Consider the function

$$F : \mathbb{S}^n \to \mathbb{R}^n, \ F(X) = (\lambda_1(X), \ldots, \lambda_n(X)), \quad (X \in \mathbb{S}^n) \tag{12}$$

where $\lambda_1(X) \geq \ldots \geq \lambda_n(X)$ are the eigenvalues of X.

[3] $g(Py) = g(y)$ for any permuation matrix P.

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Let

$$K = \left\{ v \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{k} v_i \geq 0, k = 1, \ldots, n-1, \sum_{i=1}^{n} v_i = 0 \right\}.$$

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Then

- F is K-convex;
- $\operatorname{dom} F = \mathbb{S}^n$;
- If $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and permutation invariant[3], then $g(F(X)) \leq g(y) \quad ((X, y) \in K\text{-}\operatorname{epi} F)$.

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Proposition 17 (Lewis '95/Burke, H., Nguyen '19).

Let g be proper, convex and permutation invariant and let $F : \mathbb{S}^n \to \mathbb{R}^n$ be given by (12). Then the following hold:

a) *$g \circ F$ is convex and $(g \circ F)^* = g^* \circ F$.*

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a) *$g \circ F$ is convex and $(g \circ F)^* = g^* \circ F$.*

b) *For all $X \in F^{-1}(\mathrm{dom}\, g)$ we have that*

$$\partial(g \circ F)(X) = \bigcup_{v \in \partial g(\lambda(X))} \mathrm{conv}\left\{ U^* \mathrm{diag}(v) U \,\middle|\, U^* X U = \mathrm{diag}(F(X)),\ U^* U = I_n \right\}.$$

[3] $g(Py) = g(y)$ for any permuation matrix P.

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